Symmetric Functions
On the structure constants of their multiplication

K. N. Raghavan, IMSc (HBNI)
joint with Mrigendra Singh Kushwaha and Sankaran Viswanath

reference: arXiv:1905.05302 [math.RT]
A study of Kostant-Kumar modules via Littelmann paths

22 July 2020
Bhaskaracharya Pratishthana
Professor S. S. Abhyankar's Birthday Celebrations

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Executive summary (abstract) of the talk

Symmetric polynomials are interesting, not just for their own sake. They appear naturally in various contexts: e.g., in geometry in the study of cohomology of Grassmannians and other homogeneous spaces, in the representation theory of semisimple Lie algebras such as $sl_n(\mathbb{C})$.

Consider the space Λ_n of symmetric polynomials (in n variables, over $\mathbb{C}$). It has many different (vector space) bases, perhaps the most interesting of which is the one consisting of Schur polynomials.

The question is: how do we write a product of Schur polynomials as a linear combination of Schur polynomials?

The answer is given by the Littlewood-Richardson (LR) rule. It has geometric and representation theoretic significance.

OUTLINE OF THE TALK

- What are symmetric polynomials?
- Elementary symmetric polynomials
- Parametrizing set for symmetric polynomials
- The Schur polynomials
- Multiplication of Schur polynomials: the Littlewood-Richardson (LR) rule
- joint work with M. S. Kushwaha & S. Viswanath: a refinement of the LR rule
- Concluding remarks

SYMMETRIC POLYNOMIALS: DEFINITION

Which of the following are symmetric?

- $x_1^3 + 2x_1^2x_2 + 2x_1x_2^2 + x_2^3 - 7x_1x_2$ variables: x_1, x_2
- $-x_1 - x_2 - x_3 + 7x_1^2x_2 + 7x_2^2x_3 + 7x_3^2x_1$ variables: x_1, x_2, x_3
- $-10x_1x_2x_3 + 7x_1x_2 + 7x_2x_3 + 7x_1x_3$ variables: x_1, x_2, x_3

SYMMETRIC POLYNOMIALS: DEFINITION

$x_1^3 + 2x_1^2x_2 + 2x_1x_2^2 + x_2^3 - 7x_1x_2$ is symmetric in x_1, x_2.

$-x_1 - x_2 - x_3 + 7x_1^2x_2 + 7x_2^2x_3 + 7x_3^2x_1$ not symmetric in x_1, x_2, x_3

$-10x_1x_2x_3 + 7x_1x_2 + 7x_2x_3 + 7x_3x_1$ is symmetric in x_1, x_2, x_3

HIGH SCHOOL PROBLEM

If x_1 and x_2 are the roots of the polynomial $t^2 + 2019t - 5$, what is $x_1^2 + x_2^2$?

$(t - x_1)(t - x_2) = t^2 - (x_1 + x_2)t + x_1x_2$

$x_1 + x_2 = -2019, \quad x_1x_2 = -5.$

$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (-2019)^2 - 2 \cdot (-5)$

One more high school problem

If x_1, x_2, x_3, x_4, and x_5 are the roots of the polynomial $t^5 + 2t^4 - 3t^3 + 7t - 9$, find

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = (x_1 + x_2 + x_3 + x_4 + x_5)^2 - 2 \sum_{1 \leq i < j \leq 5} x_i x_j$$

$$(t - x_1)(t - x_2)(t - x_3)(t - x_4)(t - x_5) = \\ t^5 - (x_1 + x_2 + x_3 + x_4 + x_5)t^4 + (\sum_{1 \leq i < j \leq 5} x_i x_j)t^3 - \cdots$$

Thus

$$(x_1 + x_2 + x_3 + x_4 + x_5) = -2 \qquad \& \qquad \sum_{1 \leq i < j \leq 5} x_i x_j = -3$$

and

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = (-2)^2 - 2(-3) = 10$$

Elementary symmetric functions

$$e_0 = 1$$
$$e_1 = x_1 + x_2 + \cdots + x_n$$
$$e_2 = \sum_{1 \le i < j \le n} x_i x_j$$
$$\vdots$$
$$e_n = x_1 x_2 \cdots x_n$$

These appear as coefficients in:

$$(t - x_1)(t - x_2) \cdots (t - x_n) = t^n - e_1 t^{n-1} + e_2 t^{n-2} + \cdots + (-1)^n e_n$$

Basic theorem

Theorem

Any symmetric function in n variables $x_1, \ldots, x_n$ is uniquely a polynomial in the elementary symmetric polynomials $e_1, \ldots, e_n$.

Another way of saying this:

$$\Lambda_n = \mathbb{C}[e_1, \ldots, e_n] \subseteq \mathbb{C}[x_1, \ldots, x_n]$$

and moreover there are no relations among $e_1, \ldots, e_n$.

Λ_n is isomorphic to a polynomial ring in n variables!

Parametrizing set for a basis of Λ_n

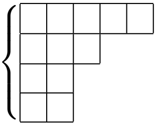
$\Lambda_n = \mathbb{C}[e_1, \ldots, e_n]$ is a graded subring of $\mathbb{C}[x_1, \ldots, x_n]$.
$\deg(e_1) = 1, \deg(e_2) = 2, \ldots, \deg(e_n) = n$.

Monomials in $e_1, \ldots, e_n$ form a basis of Λ_n
(since it is a polynomial ring in $e_0, \ldots, e_n$).

But $e_1^{m_1} \ldots e_n^{m_n}$ has degree $1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n$.

PARTITIONS AND ASSOCIATED MONOMIALS IN THE e_i

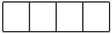
$\lambda = 5 + 3 + 2 + 2 \quad \leftrightarrow \quad$ # of parts = 4

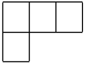


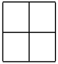
$\leftrightarrow \quad e_1^2 e_2 e_4^2$

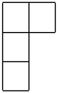
A parametrizing set for the homogeneous piece $(\Lambda_n)_d$ of degree d of Λ_n consists of partitions of d with at most n parts.

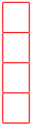
ILLUSTRATION: PARAMETRIZING SET FOR A BASIS OF $(\Lambda_3)_4$

 $\leftrightarrow e_1^4$

 $\leftrightarrow e_1^2 e_2$

 $\leftrightarrow e_2^2$

 $\leftrightarrow e_1 e_3$

 $\leftrightarrow e_4$

does NOT count having 4 parts

Would count for $(\Lambda_n)_4$ if $n \geq 4$

$\dim\,(\Lambda_3)_4 = 4 \qquad \dim\,(\Lambda_n)_4 = 5$ for $n \geq 4 \qquad \dim\,(\Lambda_2)_4 = 3$

To reiterate:

A parametrizing set for the homogeneous piece $(\Lambda_n)_d$ of degree d of Λ_n consists of partitions of d with at most n parts.

Semi-standarad Young Tableau = SSYT = A shape filled with numbers s/t ...

Fix $\lambda \vdash d$.

Example: $d = 12, \quad \lambda = 5 + 3 + 2 + 2, \quad n = 5$

1	1	2	2	3
2	2	3		
3	4			
4	5			

- Boxes of λ filled with elements of $[n] := \{1, 2, \ldots, n\}$.
- Repetitions allowed. Not every element of $[n]$ need be used.
- Entries along each row weakly increasing.
- Entries along each column strictly increasing.

Schur polynomials

Recall that partitions of d with at most n parts form a basis of $(\Lambda_n)_d$.

Given such a partition λ, we define a symmetric function $\mathfrak{s}_\lambda$ (in n variables of degree d) as follows:

Example: $d = 3, \quad \lambda = 2 + 1, \quad n = 3;$ Enumeration of the SSYT:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

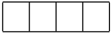
$$\mathfrak{s}_\lambda := x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

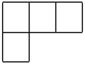
Note that $\mathfrak{s}_\lambda$ is symmetric in x_1, x_2, x_3.

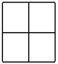
Theorem

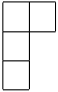
The $\mathfrak{s}_\lambda$ are symmetric in $x_1, \ldots, x_m$. Moreover, as λ varies over partitions of d with at most n parts, the $\mathfrak{s}_\lambda$ form a basis for $(\Lambda_n)_d$.

Illustration: basis of Schur polynomials for $(\Lambda_3)_4$

 $\leftrightarrow$ complete symmetric polynomial in 3 variables of degree 4

 $\leftrightarrow (x_1x_2+x_1x_3)(x_1^2+x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2)+x_2x_3(x_2^2+x_2x_3+x_3^2)$

 $\leftrightarrow x_1x_2(x_1x_2+x_1x_3+x_2x_3)+x_1x_3(x_1x_3+x_2x_3)+x_2^2x_3^2$

 $\leftrightarrow x_1x_2x_3(x_1+x_2+x_3)$

ALTERNATIVE DEFINITION OF SCHUR POLYNOMIALS

Let $n = 3$ and $\lambda = 2 + 0 + 0 \leftrightarrow \square\square$

$$\begin{aligned}\mathfrak{s}_\lambda =& x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 \\ =& \text{complete symmetric polynomial of degree 2}\end{aligned}$$

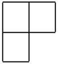
$$\mathfrak{s}_\lambda = \frac{\begin{vmatrix} x_1^{2+2} & x_2^{2+2} & x_3^{2+2} \\ x_1^{0+1} & x_2^{0+1} & x_3^{0+1} \\ x_1^{0+0} & x_2^{0+0} & x_3^{0+0} \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1^1 & x_2^1 & x_3^1 \\ x_1^0 & x_2^0 & x_3^0 \end{vmatrix}}$$

This is the **Weyl character formula**.

CHARACTERS

finite group	Present context
defined as trace	defined as trace
are class functions	are symmetric polynomials
from o.n. basis for class functions	form basis of Λ_n (even orthogonal)
determine representations	determine representations

An illustration

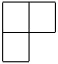
 $\leftrightarrow x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$

 $\leftrightarrow x_1 + x_2 + x_3$

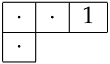
Question: $\mathfrak{s}_{2+1} \cdot \mathfrak{s}_1$ belongs to $(\Lambda_3)_4$.

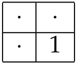
How to express it as a linear combination of the basis elements of the previous slide?

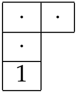
An illustration

 $\leftrightarrow x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$

 $\leftrightarrow x_1 + x_2 + x_3$

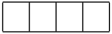
 $\leftrightarrow (x_1x_2+x_1x_3)(x_1^2+x_1x_2+x_1x_3+x_2^2+x_2x_3+x_3^2)+x_2x_3(x_2^2+x_2x_3+x_3^2)$

 $\leftrightarrow x_1x_2(x_1x_2 + x_1x_3 + x_2x_3) + x_1x_3(x_1x_3 + x_2x_3) + x_2^2x_3^2$

 $\leftrightarrow x_1x_2x_3(x_1 + x_2 + x_3)$

Check: $\mathfrak{s}_{2+1} \cdot \mathfrak{s}_1 = \mathfrak{s}_{3+1} + \mathfrak{s}_{2+2} + \mathfrak{s}_{2+1+1}$.

The following doesn't figure in the RHS since it doesn't "contain" the partition $2 + 1$.

 $\leftrightarrow$ complete symmetric polynomial in 3 variables of degree 4

A more detailed illustration

$$2+1=\begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftrightarrow \mathfrak{s}_{2+1} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Question: $\mathfrak{s}_{2+1} \cdot \mathfrak{s}_{2+1} =$? (note that it belongs to $(\Lambda_3)_6$).

Answer: $\mathfrak{s}_{2+1} \cdot \mathfrak{s}_{2+1} = \mathfrak{s}_{4+2} + \mathfrak{s}_{4+1+1} + \mathfrak{s}_{3+3} + 2\mathfrak{s}_{3+2+1} + \mathfrak{s}_{2+2+2}$

$$\begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 1 & 1 \\ \hline \cdot & 2 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline \cdot & \cdot & 1 & 1 \\ \hline \cdot & \multicolumn{3}{c}{} \\ \cline{1-1} 2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 1 \\ \hline \cdot & 1 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline \cdot & \cdot & 1 \\ \hline \cdot & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \cdot & \cdot & 1 \\ \hline \cdot & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline \cdot & \cdot \\ \hline \cdot & 1 \\ \hline 1 & 2 \\ \hline\end{array}$$

Refinement of the LR rule

Representation theoretically, LR rule tells us how to compute the coefficients $c^{\nu}_{\lambda\mu}$ in:

$$V_\lambda \otimes V_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} V_\nu$$

There is a natural filtration of $V_\lambda \otimes V_\mu$ by the "Kostant-Kumar" submodules $U\mathfrak{g}(v_\lambda \otimes v_{w\mu})$ as w varies over the Weyl group (symmetric group).

How do the Kostant-Kumar submodules decompose?

There is a way to assign elements of the Weyl group to LR components so that we can get an answer to this question.

CONCLUDING REMARKS

- The Lie algebraic L-R rule contains the L-R rule for symmetric functions as a special case.
- There are *Schubert polynomials* that generalize Schur polynomials.

  The analogous problem (determining their multiplication) called **Schubert Calculus** is open.

Thank you for your attention.