A Brief Introduction to Modular Forms

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1. Introduction

In this article, a brief introduction to the subject of modular forms is being given. For most of the statements I have not given the proofs. There are several good books written on this subject and I have listed some of them in the references. The interested reader can look at the books for the details of proofs. I have also given some articles related to the theory of newforms (which has been presented at the end). I hope that the reader will have some feeling for the subject from these notes. I would like to thank S. D. Adhikari and S. A. Katre for a careful reading of the manuscript.

2. Modular Forms over $SL_2(\mathbb{Z})$

Let k be a rational integer and let $\mathcal{H}$ denote the Poincaré upper half-plane, consisting of complex numbers z with positive imaginary part. Let $M_2(R)$ denote the set of all 2×2 matrices whose entries lie in the ring R. The *modular group*, denoted by $SL_2(\mathbb{Z})$ is the group defined as follows.

$$SL_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \middle| ad - bc = 1 \right\}.$$

It is a discrete subgroup of

$$GL_2^+(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{R}) \middle| ad - bc > 0 \right\}.$$

$GL_2^+(\mathbb{R})$ acts on $\mathcal{H}$ as follows. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ and $z \in \mathcal{H}$,

$$\gamma \cdot z = \frac{az+b}{cz+d}.$$

This is called the Möbius transformation. It is an easy exercise to check that

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

So, if $z \in \mathcal{H}$ and $\gamma \in GL_2^+(\mathbb{R})$, then $\gamma z \in \mathcal{H}$. i.e., $\mathcal{H}$ is preserved by the Möbius transformation. We also extend the action of $\gamma \in GL_2^+(\mathbb{R})$ to $\mathbb{Q} \setminus \{-d/c\}$ by the same rule.

Further, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ we define

$$\gamma \cdot \infty = \frac{a}{c} \quad \text{and } \gamma \cdot (\frac{-d}{c}) = \infty,$$

So, we have the action of $GL_2^+(\mathbb{R})$ on $\mathcal{H} \cup \{\infty\} \cup \mathbb{Q}$.

We first give the formal definition of a modular forms below and the explanations shall follow the definition.

DEFINITION 2.1. *A function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be a modular function (resp. form) of weight k for the full modular group $SL_2(\mathbb{Z})$, if it satisfies the following:*

(i) *f is a meromorphic (resp. analytic) function.*
(ii) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.
(iii) *$f(z)$ has the following Fourier expansion.*

$$f(z) = \sum_{n=-m_0}^{\infty} a(n) q^n \qquad (q = e^{2\pi i z}), m_0 > 0$$

$$\left(\text{resp.} \quad f(z) = \sum_{n=0}^{\infty} a(n) q^n\right).$$

A modular form is said to be a cusp form if further $a(0) = 0$ in (iii) *above.*

Notation. The set of all modular forms (resp. cusp forms) of weight k for the group $SL_2(\mathbb{Z})$ is denoted by M_k (resp. S_k).

The condition (ii) above can be written as:

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) =: f\Big|_k \gamma(z) = f(z) \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

In terms of this new stroke notation, we want the following.

$$f\Big|_k \gamma_1 \Big|_k \gamma_2(z) = f\Big|_k \gamma_1\gamma_2(z).$$

In other words, defining $j(\gamma, z) = cz + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, we want the equality

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2\ z) \cdot j(\gamma_2, z),$$

which can easily be verified. The factor $j(\gamma, z)$ is called the *automorphy factor*. Let us now analyze the condition (iii). Consider the change of variable

$$z \longmapsto e^{2\pi i z} =: q.$$

This takes the Poincaré upper half-plane to the punctured open disk centred at the origin. We agree to take the point at ∞ to the origin under this map. The real line maps onto the boundary of the disk. Since we have the condition that f is meromorphic on $\mathcal{H}$, under this transformation, f has a Laurent expansion around the origin, say

$$f(z) = \sum_{-\infty}^{\infty} b(n) q^n.$$

We say that f is *meromorphic* at ∞ (resp. *holomorphic* at ∞) if $b(n) = 0$ for $n \ll 0$ (resp. for $n < 0$). Here $n \ll 0$ means that only finitely many coefficients $b(n)$ for $n < 0$ can be non-zero. So the condition (iii) is equivalent to saying that f is *meromorphic* (resp. *holomorphic*) at ∞.

Cusps. The points $\mathbb{Q} \cup \{\infty\}$ are called the *cusps*. Let $s \in \mathbb{Q}$. Then, s can be written in the reduced form $s = \frac{a}{c}$, where $\gcd(a, c) = 1$. Now complete a, c to get the matrix $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Then it is clear that $s = \gamma \cdot \infty$. But one has a natural equivalence (modulo the group $SL_2(\mathbb{Z})$) among the cusps, namely, the cusps s_1 and s_2 are said to be $SL_2(\mathbb{Z})$ equivalent if there exists a matrix $\gamma \in SL_2(\mathbb{Z})$ satisfying $s_1 = \gamma \cdot s_2$. From the above remarks. it is clear that all rational numbers are $SL_2(\mathbb{Z})$ equivalent to ∞. This is the reason why we have only one Fourier expansion for f.

Fundamental domain. The action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ divides $\mathcal{H}$ into equivalence classes, called *orbits*. Selecting one point from each orbit one gets a *fundamental set* for $SL_2(\mathbb{Z})$. Modifying the concept slightly for having nice topological properties, one obtains what are called *fundamental domains*. A fundamental domain for $SL_2(\mathbb{Z})$ is given as follows.

$$\mathcal{F} = \left\{ z \in \mathcal{H} \;\middle|\; |z| \geq 1, \frac{-1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \right\}. \tag{1}$$

The equivalence is defined as usual: two points z_1 and z_2 are said to be $SL_2(\mathbb{Z})$- equivalent, if there exists a matrix γ belonging to $SL_2(\mathbb{Z})$ which takes one to the other. The fact that $\mathcal{F}$ is a fundamental domain for $SL_2(\mathbb{Z})$ is equivalent to the following:

(i) For any $z \in \mathcal{H}$, there exists a unique z_1 belonging to $\mathcal{F}$ such that $z = \gamma z_1$, for some $\gamma \in SL_2(\mathbb{Z})$.

(ii) No two interior points of $\mathcal{F}$ are $SL_2(\mathbb{Z})$ equivalent. (If we take the fundamental domain suitably, one can omit the condition "interior".)

Let us briefly indicate the proof of the fact that $\mathcal{F}$ is a fundamental domain for $SL_2(\mathbb{Z})$. The modular group $SL_2(\mathbb{Z})$ contains the following two special matrices.

$$T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right); \qquad Tz = z + 1 \quad \text{(translation.)}$$
$$S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right); \qquad Sz = -1/z \quad \text{(inversion.)}$$

For a given point $z \in \mathcal{H}$, the idea is to apply the translations T^m ($m \in \mathbb{Z}$ to get a point, say z_1, inside the strip $-1/2 \leq \mathrm{Re}(z_1) \leq 1/2$. If this point z_1 is inside $\mathcal{F}$, then we are through. Otherwise, apply the inversion map S to z_1 to get another point z_2, which will lie outside the unit circle. The process is continued till we get a point inside our region $\mathcal{F}$. (It is a fact that this process ends after a finite number of steps!) This proof implies an interesting fact that the modular group $SL_2(\mathbb{Z})$ is generated by the two elements T and S.

Weight formula. Let f be a complex valued meromorphic function defined on $\mathcal{H}$. For a point $P \in \mathcal{H}$, let $v_P(f)$ denote the order of zero of f at P and $v_\infty(f)$ denote the least integer n for which $a(n)$ is non-zero in the expansion (iii) of Definition 2.1. Then we have the following theorem.

THEOREM 2.1. *Let f be a non-zero modular function of weight k for the group $SL_2(\mathbb{Z})$. Then*

$$v_\infty(f) + \frac{1}{3}v_\rho(f) + \frac{1}{2}v_i(f) + \sum_{\substack{P \in \mathcal{H} \setminus SL_2(\mathbb{Z}) \\ P \neq i, \rho}} v_P(f) = \frac{k}{12}, \tag{2}$$

where $\rho = -1/2 + i\sqrt{3}/2$.

The above theorem is proved by integrating the function f'/f along a specific contour in $\mathcal{H}$ and using the residue theorem. We omit the details here. This theorem has many applications as we shall see in the sequel.

REMARK 2.1. (1) There is no modular form of weight $k < 0$. (This is clear from the formula (2) as the left hand side is always non-negative.)

(2) There is no non-zero modular form of odd weight. This will follow directly using Definition 2.1. Take the matrix $\gamma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. By the condition (ii) of Definition 2.1, we must have, $(-1)^k f(z) = f\Big|_k \gamma(z) = f(z)$. If k is odd, it follows that $f(z) = 0$.

(3) If f is a modular form of weight $k = 0$, then $f \in \mathbb{C}$. Assume that $f \neq 0$ is a modular form of weight 0. Then by weight formula, f does not vanish on $\mathcal{H} \cup \{\infty\}$. Put $c = v_\infty(f)$. Then, $g = f - c$ is a modular form of weight 0 and it vanishes at ∞ (by definition), This implies that $g = 0$. Therefore, $f = c \in \mathbb{C}$.

(4) Let $k = 2$. Then all the terms on the left hand side of (2) are non-negative. So, we must have $f = 0$.

From the above remark, it follows that the first non-trivial example of a modular form occurs only when $k \geq 4$. We shall present now an example of a modular form, which plays an important role in the theory of modular forms.

Eisenstein series. For $k \geq 4$, and $z \in \mathcal{H}$, put

$$G_k(z) = \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} (mz+n)^{-k} \tag{3}$$

We will show that $G_k(z) \in M_k$.
Since $k \geq 4$, the series on the right hand side of (3) converges absolutely and uniformly on any compact subset of $\mathcal{H}$. Therefore $G_k(z)$ defines a holomorphic function on $\mathcal{H}$. Now consider

$$\begin{aligned}\lim_{z\to i\infty} G_k(z) &= \lim_{z\to i\infty} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} (mz+n)^{-k}\\ &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} n^{-k} + \lim_{z\to i\infty} \sum_{\substack{m,n\in\mathbb{Z}\\ m\neq 0}} (mz+n)^{-k}\\ &= 2\sum_{n\geq 1} n^{-k}\\ &= 2\zeta(k),\end{aligned} \tag{4}$$

where $\zeta(s) = \sum_{n\geq 1} n^{-s}$ is the well known Riemann zeta function. Since the limit exists, it is clear that $G_k(z)$ has no negative term in its Fourier expansion and in fact, we have shown that the constant term in the Fourier expansion is $2\zeta(k)$. This implies that $G_k(z)$ is holomorphic at $i\infty$. It remains to prove that $G_k(z)$ is invariant under $SL_2(\mathbb{Z})$ with respect to the stroke operator $\Big|_k$.

$$\begin{aligned}G_k\Big|_k T(z) &= \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} (m(z+1)+n)^{-k}\\ &= \sum_{\substack{m,n'\in\mathbb{Z}\\(m,n')\neq(0,0)}} (mz+n')^{-k}\\ &= G_k(z).\end{aligned} \tag{5}$$

Next we consider the transformation with respect to S.

$$\begin{aligned}G_k\Big|_k S(z) &= z^{-k} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} (m(-1/z)+n)^{-k}\\ &= \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} (-m+nz)^{-k}\\ &= G_k(z).\end{aligned} \tag{6}$$

(In the last line, we have used the fact that the series converges absolutely.) Since S and T generate $SL_2(\mathbb{Z})$, it follows that $G_k(z)$ is invariant under $SL_2(\mathbb{Z})$ with respect to the stroke operation. We have thus established the fact that $G_k(z)$ is a modular form of weight k for the group $SL_2(\mathbb{Z})$.

Let us now derive the Fourier expansion of $G_k(z)$.

First let us prove the following lemma which is needed in getting the Fourier expansion of $G_k(z)$.

LEMMA 2.2.

$$\zeta(k) = \frac{-(2i\pi)^k}{2(k!)} B_k, \tag{7}$$

where B_k denotes the $k-th$ Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

PROOF. The product formula for sine function is

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right).$$

Taking logarithmic derivative with respect to z,

$$\begin{aligned} \pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) \qquad (8) \\ \text{i. e.,} \quad \pi z \cot \pi z &= 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2}. \end{aligned}$$

Substituting $2i\pi z = x$ on the right hand side of the above equation, we get,

$$\begin{aligned} \pi z \cot \pi z &= 1 + 2\sum_{n=1}^{\infty} \frac{x^2}{x^2 + (2\pi n)^2} \\ &= 1 + 2\sum_{n=1}^{\infty} \left(\frac{x}{2\pi n}\right)^2 \left(1 + \left(\frac{x}{2\pi n}\right)^2\right)^{-1} \\ &= 1 + 2\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} \left(\frac{x}{2\pi n}\right)^{2m} \\ &= 1 + 2\sum_{m=1}^{\infty} \zeta(2m) \left(\frac{x}{2\pi}\right)^{2m}. \end{aligned} \tag{9}$$

On the other hand,

$$\begin{aligned} \pi z \cot \pi z = \pi z \frac{\cos \pi z}{\sin \pi z} \quad &= i\pi z \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \\ = i\pi z \frac{e^{2i\pi z} + 1}{e^{2i\pi z} - 1} \quad &= i\pi z + \frac{2i\pi z}{e^{2i\pi z} - 1} \\ = \frac{x}{2} + \frac{x}{e^x - 1} \quad &= \frac{x}{2} + \sum_{\ell=0}^{\infty} \frac{B_\ell}{\ell!} x^\ell. \end{aligned} \tag{10}$$

Comparing the k-th power of x in (9) and (10), we get the required result. □

Proposition 2.3.

$$G_k(z) = 2\zeta(k)\left[1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n\right]. \tag{11}$$

In the above, $\sigma_r(n) = \sum_{d|n} d^r$.

Proof. As in the proof of the above lemma, we have,

$$\begin{aligned}\pi\cot\pi z &= i\pi\left(1 - \frac{2}{1-e^{2i\pi z}}\right)\\ &= i\pi\left(1 - 2\sum_{n=0}^{\infty} e^{2i\pi nz}\right)\\ &= -i\pi\left(1 + 2\sum_{n=1}^{\infty} q^n\right).\end{aligned} \tag{12}$$

Recalling (8), we have,

$$\begin{aligned}\pi\cot\pi z &= \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right)\\ &= \frac{1}{z} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)\end{aligned} \tag{13}$$

Comparing (12) and (13), we have,

$$\frac{1}{z} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) = -i\pi\left(1 + 2\sum_{n=1}^{\infty} q^n\right). \tag{14}$$

Differentiating the above (with respect to z) $k-1$ times, we get,

$$(k-1)!\sum_{n=-\infty}^{\infty}(z+n)^{-k} = (2i\pi)^k\sum_{n=1}^{\infty} n^{k-1}q^n. \tag{15}$$

Using the definition of $G_k(z)$, we have,

$$\begin{aligned} G_k(z) &= 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}(mz+n)^{-k} \\ &= 2\zeta(k) + 2\frac{(2i\pi)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1}q^{mn} \qquad \text{(using (15))} \\ &= 2\zeta(k) - \frac{4k}{B_k}\zeta(k)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{k-1}q^{mn} \text{ (using Lemma 2.2)} \\ &= 2\zeta(k)\left[1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n\right]. \end{aligned}$$

This completes the proof. □

We put

$$E_k(z) = \frac{1}{2\zeta(k)}G_k(z).$$

Then from the above Proposition, the Fourier expansion of E_k ($k \geq 4$) can be written as

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n. \tag{16}$$

The Eisenstein series $E_k(z)$ is called the *normalised* Eisenstein series.

Before we move onto another important example of a modular form (in fact, a cusp form), we remark about some properties of modular forms.

Remark 2.2.

(1) If $f \in M_k$, then λf also belongs to M_k, for every $\lambda \in \mathbb{C}$.
(2) If $f, g \in M_k$, then $f + g \in M_k$.
(3) If $f \in M_{k_1}$ and $g \in M_{k_2}$, then $fg \in M_{k_1+k_2}$.
(4) If f and g are the same as in the previous case with $g \neq 0$, then f/g is a modular function of weight $k_1 - k_2$.
(5) The above facts imply that M_k is a $\mathbb{C}$ -vector space.

Using the weight formula (2), one can arrive at the conclusion that the $\mathbb{C}$ -vector space M_k is one-dimensional for $4 \leq k \leq 10$, and in these cases M_k is generated by the Eisenstein series G_k. So, there is no cusp form of weight $k \leq 10$. The first example of a cusp form occurs when $k = 12$.

An example of a cusp form. Put

$$\Delta(z) = \frac{1}{1728}\left(E_4(z)^3 - E_6(z)^2\right). \tag{17}$$

Since $E_4^3(z)$ and $E_6^2(z)$ belong to M_{12}, $\Delta(z)$ is a modular form of weight 12 for $SL_2(\mathbb{Z})$. Using the Fourier expansion of $E_k(z)$ given by (16), we get

$$E_4(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$$
$$E_6(z) = 1 - 504\sum_{n\geq 1}\sigma_5(n)q^n$$

and so, we have

$$E_4^3(z) = 1 + 720\, q + 179280\, q^2 + \cdots$$
$$E_6^2(z) = 1 - 1008\, q - 220752\, q^2 + \cdots.$$

Therefore,

$$\begin{aligned}\Delta(z) &= \frac{1}{1728}\left(E_4(z)^3 - E_6(z)^2\right)\\ &= \frac{1}{1728}\left(1728q - 41472q^2 + \cdots\right)\\ &= \sum_{n=1}^{\infty}\tau(n)q^n,\end{aligned} \tag{18}$$

and hence $\Delta(z)$ is a cusp form of weight 12 for $SL_2(\mathbb{Z})$.
In the above, $\tau(n)$ is called the Ramanujan's tau function. As remarked above, $\Delta(z)$ is the first example of a cusp form. S. Ramanujan is the first mathematician to notice some nice arithmetical properties of the function $\tau(n)$. From the definition it is clear that $\tau(n) \in \mathbb{Z}\ \forall\ n$.

The first few values of $\tau(n)$ are: $\tau(1) = 1, \tau(2) = -24, \tau(3) = 252$, $\tau(4) = -1472$, $\tau(5) = 4830$, $\tau(6) = -6048$, $\tau(7) = -16744$, $\tau(8) = 84480$, $\tau(9) = -113643$, $\tau(10) = -115920$, $\tau(11) = 534612$, $\tau(12) = -370942, \cdots$. There is a table of $\tau(n)$'s for $n \leq 1000$ given by G. N. Watson. The interested reader can refer to [12]. There is a remarkable product formula for $\Delta(z)$ (due to Jacobi), which we shall give below without proof.

$$\Delta(z) = \sum_{n\geq 1}\tau(n)q^n = q\prod_{n\geq 1}(1-q^n)^{24} = (\eta(z))^{24}. \tag{19}$$

(In the above, $\eta(z)$ is the Dedekind eta-function.) The Ramanujan function $\tau(n)$ satisfies the following remarkable congruence:

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691} \qquad \text{for all } n \geq 1. \tag{20}$$

There are some other congruences involving $\tau(n)$. But we shall not go into the details here.

REMARK 2.3. The space M_k is a finite dimensional vector space. Moreover, it is generated by the Eisenstein series $E_4(z)$ and $E_6(z)$. That is, given any modular form $f(z)$ in M_k, it can be expressed as a linear combination of E_4 and E_6. More precisely,

$$f(z) = \sum_{\substack{0\leq a,b\in\mathbb{Z}\\ 4a+6b=k}} c_{a,b}\ E_4(z)^a E_6(z)^b, \tag{21}$$

where $c_{a,b} \in \mathbb{C}$. Further, one has:

$$M_k = \mathbb{C}E_k \bigoplus S_k. \tag{22}$$

In fact, multiplication by the discriminant function $\Delta(z)$ gives an isomorphism between M_{k-12} and M_k. Using this one has the following dimension formula.

$$\dim_{\mathbb{C}} M_k = \begin{cases} \left[\frac{k}{12}\right] & if\ k \equiv 2 \pmod{12} \\ 1 + \left[\frac{k}{12}\right] & if\ k \not\equiv 2 \pmod{12} \end{cases} \tag{23}$$

The modular invariant $j(z)$.

The Klein's modular invariant is defined as follows:

$$j(z) = 1728\frac{E_4(z)^3}{E_4(z)^3 - E_6(z)^2} = \frac{E_4(z)^3}{\Delta(z)}. \tag{24}$$

Since $E_4(z)^3$ and $\Delta(z)$ are modular forms of the same weight (weight 12) for $SL_2(\mathbb{Z})$, by Remark 2.2, $j(z)$ is a modular function of weight 0 for $SL_2(\mathbb{Z})$. Since it is a modular function of weight 0 for $SL_2(\mathbb{Z})$, from the definition it follows that $j(z)$ is invariant under $SL_2(\mathbb{Z})$. That is

$$j\left(\frac{az+b}{cz+d}\right) = j(z) \text{ for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}).$$

(That is why this function is called the *modular invariant*.) This function was studied extensively by F. Klein.

By the weight formula, it can be seen that

$$\Delta(z) \neq 0 \qquad \text{for all } z \in \mathcal{H}.$$

Further, $\Delta(z)$ vanishes only at ∞. Therefore, by the definition, $j(z)$ is a holomorphic function on $\mathcal{H}$ and it has a simple pole at ∞. It has the following Fourier expansion.

$$j(z) = q^{-1} + 744 + \sum_{n\geq 1} c(n)\ q^n. \tag{25}$$

The coefficients $c(n)$'s are integers. These coefficients also satisfy some nice congruence properties. Here again, we shall not go into the details. In the following remark, we shall mention some of the main properties of $j(z)$.

Remark 2.4.

1. The modular function j defines a bijection of $\mathcal{H}/SL_2(\mathbb{Z})$ onto $\mathbb{C}$.
2. The following are equivalent:
   (a) f is a modular function of weight 0 for $SL_2(\mathbb{Z})$.
   (b) f is a quotient of two modular forms of the same weight for $SL_2(\mathbb{Z})$.
   (c) f is a rational function of $j(z)$.

Hecke theory. In 1916, S. Ramanujan conjectured the following properties satisfied by the Ramanujan function $\tau(n)$.

(i) $\tau(n)$ is a multiplicative function. i.e.,

$$\tau(mn) = \tau(m)\tau(n) \quad \text{if } \gcd(m,n) = 1. \tag{26}$$

(ii) $|\tau(p)| \leq 2p^{11/2}$.

Both of these conjectures have been proved; the first one by L. J. Mordell in 1917 and the second one by P. Deligne in 1973. The proof of conjecture (i) is the starting point of the theory of Hecke operators. Here, we shall briefly explain the Hecke theory for modular forms of integral weight.

$$\text{Put} \quad \Delta_n = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \middle| ad - bc = n \right\}. \tag{27}$$

Now, define an equivalence relation on Δ_n as follows. Two elements γ_1, γ_2 of Δ_n are said to be equivalent (modulo $SL_2(\mathbb{Z})$) if and only if $\gamma_1\gamma_2^{-1} \in SL_2(\mathbb{Z})$. It can be easily seen that this is an equivalence relation. So, one has the following decomposition for Δ_n:

$$\Delta_n = \cup_i SL_2(\mathbb{Z})\gamma_i, \tag{28}$$

where γ_i's are a finite number of representatives for the equivalence classes. Define the n-th Hecke operator by

$$f\Big|T_n(z) := n^{\frac{k}{2}-1} \sum_i f\Big|_k \gamma_i \qquad (f \in M_k). \tag{29}$$

(Here $f\Big|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) := (ad-bc)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$.)

In our case, the exact set of representatives γ_i's are given as follows:

$$\left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \middle| a, b, d \in \mathbb{Z} \text{ such that } ad = n, 0 \leq b < d \right\}. \tag{30}$$

When $n = p$, a prime, and for $f \in M_k$, one has:

$$\begin{aligned} f\Big|T_p &= p^{\frac{k}{2}-1} \sum_{\substack{ad=p \\ 0\le b<d}} f\Big|_k \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \\ &= p^{k-1} f(pz) + p^{-1} \sum_{0\le b<d} f\left(\frac{z+b}{p}\right). \end{aligned} \tag{31}$$

If $f(z) = \sum_{n\ge 0} a_f(n)\ q^n$, then it is easily seen that

$$f\Big|T_p(z) = \sum_{n\ge 0} b(n)\ q^n, \tag{32}$$

where

$$b(n) = a_f(np) + p^{k-1} a_f(n/p) \qquad (n \ge 0). \tag{33}$$

(Notation: $a_f(m)$ denotes the m-th Fourier coefficient of f and it is defined to be zero when m is not an integer.)

From the work of H. Petersson, there is an inner product defined in the space M_k. Let $f, g \in M_k$ with f or g a cusp form. Then the Petersson inner product of f and g is defined by

$$\langle f, g\rangle = \int_{\mathcal{F}} f(z)\overline{g(z)} y^{k-2} dxdy \qquad (z = x+iy). \tag{34}$$

The requirement that f or g to be a cusp form is needed for the convergence of the integral.

The Hecke operators T_p for all primes p are hermitian with respect to the Petersson scalar product. Further, they form a commuting family of operators. Therefore, we have the following theorem from linear algebra.

Theorem 2.4. *The vector space S_k has a basis of simultaneous eigenforms with respect to all Hecke operators T_n.*

Remark 2.5. Note that the basis constructed in Theorem 2.4 is orthogonal.

Further, it can be shown that the Hecke operators satisfy the following commuting property in general.

$$T_m T_n = \sum_{d|\gcd(m,n)} d^{k-1} T_{mn/d^2}. \tag{35}$$

Therefore, for a Hecke eigenform (this means an eigenfunction with respect to all Hecke operators T_n) $f \in S_k$, we have,

$$a_f(m)a_f(n) = \sum_{d|\,\gcd(m,n)} d^{k-1} a_f(mn/d^2). \tag{36}$$

Hence, $a_f(m)a_f(n) = a_f(mn)$ if $\gcd(m,n) = 1$.
Since S_{12} is one-dimensional, $\Delta(z)$ is a Hecke eigenform and hence by the above observation, we get

$$\tau(m)\tau(n) = \tau(mn) \quad \text{if } \gcd(m,n) = 1, \tag{37}$$

which proves the conjecture (i) of Ramanujan mentioned above.

THEOREM 2.5. *Let $k \geq 4$ and let $f = \sum_{n\geq 0} a_f(n)q^n \in M_k$ be a Hecke eigenform. Then $a_f(1) \neq 0$. In other words, the function f can be normalised.*

PROOF. Since f is a Hecke eigenform, the following is true for all primes p and for all $n \geq 0$:

$$a_f(np) + p^{k-1} a_f(n/p) = \lambda_p a_f(n), \tag{38}$$

where λ_p is the eigenvalue.

If $a_f(1) = 0$, then it follows from the above expression that,

$$a_f(p) = 0 \quad \text{for all primes } p \tag{39}$$

which implies that $a_f(n) = 0$ for all $n \geq 1$.
Therefore, $f(z) = a_f(0) \in \mathbb{C}$. Since $k \geq 4$, $a_f(0) = 0$, and so we have, $f = 0$, a contradiction. Therefore, $a_f(1) \neq 0$. □

DEFINITION 2.2. *A form $f \in M_k$ is said to be normalised, if the leading term in its Fourier expansion is equal to 1.*

REMARK 2.6. We can find an orthogonal basis of S_k consisting of normalised Hecke eigenforms.

THEOREM 2.6. *Let f be a normalised Hecke eigenform belonging to S_k. Then the eigenvalue of f for T_p is $a_f(p)$. Further, we have*

$$a_f(m)a_f(n) = \sum_{d|\,\gcd(m,n)} d^{k-1} a_f\left(mn/d^2\right) \tag{40}$$

if and only if f is a normalised Hecke eigenform in S_k.

PROOF. If f is a Hecke eigenform with eigenvalue λ_p for T_p, then we have

$$a_f(np) + p^{k-1} a_f(n/p) = \lambda_p a_f(n) \qquad \text{for all } n \geq 0. \tag{41}$$

Substituting $n = 1$ in the above and using $a_f(1) = 1$, we have $\lambda_p = a_f(p)$.

We shall now prove the second statement. First assume that

$$a_f(m)a_f(n) = \sum_{d|\gcd(m,n)} d^{k-1} a_f\left(mn/d^2\right) \quad \forall\ m,n \geq 1. \tag{42}$$

Taking $m = p$ in the above equation we have,

$$a_f(np) + p^{k-1}a_f(n/p) = a_f(p)a_f(n) \qquad \forall\ p,n \geq 1$$

i.e.,

$$f\Big|T_p = a_f(p)f \qquad \forall\ p.$$

Since $a_f(1) = 1$ (by putting $n = 1$, $m = p$), we see that f is a normalised Hecke eigenform.
Conversely, let $f \in S_k$ be a normalised Hecke eigenform. Then, we have

$$a_f(np) + p^{k-1}a_f(n/p) = a_f(p)a_f(n) \qquad \forall\ p,n \geq 1$$

i.e.,

$$a_f(p^{n+1}) = a_f(p)a_f(p^n) - p^{k-1}a_f(p^{n-1}), \quad n \geq 1$$

and

$$a_f(np^\alpha) = a_f(p)a_f(np^{\alpha-1}) - p^{k-1}a_f(np^{\alpha-2}), \alpha \geq 2, \gcd(n,p) = 1.$$

From this we conclude that

$$a_f(m)a_f(n) = a_f(mn) \quad \text{if } \gcd(m,n) = 1.$$

The general identity can be easily proved and we leave it to the reader to verify. □

Remark 2.7. It can be easily seen that

$$\sigma_k(m)\sigma_k(n) = \sum_{d|\gcd(m,n)} d^k \sigma_k\left(mn/d^2\right).$$

To prove this one has to use the following.

$$\begin{aligned} \sigma_k(p^{n+1}) &= \sigma_k(p)\sigma_k(p^n) - p^k\sigma_k(p^{n-1}) \quad n \geq 1, \\ \sigma_k(mn) &= \sigma_k(m)\sigma_k(n) \quad \text{if } \gcd(m,n) = 1. \end{aligned} \tag{43}$$

Theorem 2.7. *Let $f, g \in S_k$ be two normalised Hecke eigenforms whose eigenvalues with respect to the Hecke operators T_p are equal. Then $f = g$.*

(Note: The above theorem is often referred to as the "multiplicity 1" theorem.)

PROOF. We have proved that if f is a normalised Hecke eigenform, then the eigenvalue with respect to the Hecke operator T_p is the p-th Fourier coefficient $a_f(p)$ of f in the q-expansion. If f and g have the same eigenvalues for all T_p, then we have,

$$a_f(p) = a_g(p) \quad \forall\ p.$$

Since $a_f(1) = a_g(1) = 1$, we must have,

$$a_f(n) = a_g(n) \quad \forall\ n \geq 1.$$

This completes the proof. □

L-functions associated to modular forms. We have the following theorem of Hecke on the estimates for the Fourier coefficients of modular forms.

THEOREM 2.8. *(Hecke) Let $f \in S_k$. Then, we have*

$$a_f(n) = O(n^{k/2}).$$

(In other words, the quotient $\frac{|a_f(n)|}{n^{k/2}}$ remains bounded when $n \to \infty$.)

As a consequence, we have the following corollary.

COROLLARY 2.9. *If $f \in M_k$ and f is not a cusp form, then $a_f(n) = O(n^{k-1})$.*

REMARK 2.8. The exponent $k/2$ of the above theorem can be improved. In fact, the Ramanujan-Petersson conjecture says (Ramanujan for the weight $k = 12$ and Petersson for general k) that if $f \in S_k$, is a normalised Hecke eigenform, then

$$a_f(n) = O\left(n^{k/2-1/2}\sigma_0(n)\right).$$

This implies that $a_f(n) = O(n^{k/2-1/2+\epsilon})$ for every $\epsilon > 0$. As mentioned before, this conjecture was proved by P. Deligne as a consequences of the "Weil conjectures" about algebraic varieties over finite fields.

Let $f \in M_k$. Then the Dirichlet series associated to f is defined by

$$L_f(s) = \sum_{n \geq 1} a_f(n) n^{-s}. \tag{44}$$

We have seen that $a_f(n) = O(n^{k/2})$ if f is a cusp form and $a_f(n) = O(n^{k-1})$ if f is not a cusp form. Therefore, the Dirichlet series defined by (44) converges absolutely for Re $(s) > k/2 + 1$ if f is a cusp form and for Re $(s) > k$ if f is not a cusp form. E. Hecke found a remarkable connection between each modular form and its associated Dirichlet series.

THEOREM 2.10. *(Hecke) If the Fourier coefficients $a_f(n)$ satisfy the multiplicative property*

$$a_f(m)a_f(n) = \sum_{d|\gcd(m,n)} d^{k-1} a_f(mn/d^2), \tag{45}$$

then the Dirichlet series $L_f(s)$ has an Euler product expansion of the form

$$L_f(s) = \prod_p \left(1 - a_f(p)p^{-s} + p^{k-1-2s}\right)^{-1}, \tag{46}$$

absolutely convergent with the Dirichlet series.

REMARK 2.9. For the Ramanujan's discriminant function, we have the Euler product representation as follows.

$$\sum_{n=1}^{\infty} \tau(n)n^{-s} = \prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1} \quad \text{for Re } (s) > 7.$$

Hecke also deduced the following analytic properties of $L_f(s)$.

THEOREM 2.11. *(Hecke) Assume that $k \geq 4$. Let $L_f(s)$ be the Dirichlet series associated to the modular form $f \in M_k$, which is defined for* Re $(s) > k$. *Then, $L_f(s)$ can be continued analytically beyond the line* Re $(s) = k$ *with the following properties.*

(i) *If $a_f(0) = 0$, $L_f(s)$ is an analytic function of s.*
(ii) *If $a_f(0) \neq 0$, $L_f(s)$ is analytic for all s except for a simple pole at $s = k$ with residue*

$$\frac{(-1)^{k/2} a_f(0)(2\pi)^k}{\Gamma(s)}.$$

(iii) *The function $L_f(s)$ satisfies the functional equation*

$$(2\pi)^{-s}\Gamma(s)L_f(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)L_f(k-s). \tag{47}$$

REMARK 2.10. Hecke also proved a converse to the above theorem. He proved that every Dirichlet series which satisfies a functional equation of the type stated in the theorem together with some analytic and growth conditions necessarily arises from a modular form in M_k.

REMARK 2.11. Let f be a modular form in M_k. Then f is a normalised Hecke eigenform if and only if the associated Dirichlet series $L_f(s)$ has an Euler product of the form

$$L_f(s) = \prod_p \left(1 - a_f(p)p^{-s} + p^{k-1-2s}\right)^{-1}.$$

3. Modular Forms of Higher Level

Let $N \in \mathbb{N}$. The subgroup

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\} \tag{48}$$

of $SL_2(\mathbb{Z})$ is called the *principal* congruence subgroup of level N.

Definition 3.1. *A subgroup Γ' of $SL_2(\mathbb{Z})$ is called a congruence subgroup, if it contains $\Gamma(N)$ for some N. The least N for which $\Gamma(N) \subset \Gamma'$ is called the level of the congruence subgroup.*

Examples.

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\}. \tag{49}$$

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \middle| \; c \equiv 0 \pmod{N} \right\}. \tag{50}$$

Definition 3.2. *A function $f : \mathcal{H} \longrightarrow \mathbb{C}$ is said to be a modular form of weight k for $\Gamma_0(N)$ with character χ (χ is a Dirichlet character modulo N), if*

(i) *f is an analytic function.*
(ii) $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$.
(iii) *for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, the function $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ has a Fourier expansion of the form*

$$\sum_{n \geq 0} b_\gamma(n) q_N^n \qquad (q_N = e^{2\pi i z/N}).$$

As before, (iii) is equivalent to saying that f is holomorphic at all cusps of $\Gamma_0(N)$. Let us elaborate a little bit more. $\Gamma_0(N)$ is a subgroup of finite index in $SL_2(\mathbb{Z})$. Let $[SL_2(\mathbb{Z}) : \Gamma_0(N)] = \mu$. Then $SL_2(\mathbb{Z})$ can be written as

$$SL_2(\mathbb{Z}) = \cup_{i=1}^{\mu} \Gamma_0(N)\gamma_i.$$

Then, the set $\left\{\gamma_i\infty \middle| 1 \leq i \leq \mu\right\}$ contains the set of inequivalent cusps modulo $\Gamma_0(N)$. Let s be a cusp belonging to $\mathbb{Q}$. Let $\gamma_0 \in SL_2(\mathbb{Z})$ be such that $\gamma_0\infty = s$. Let f be a function satisfying the condition (ii) stated in the above definition. Put $g = f\Big|_k \gamma_0$. Then, it can be checked that g satisfies the condition (ii) with respect to the group $\gamma_0^{-1}\Gamma_0(N)\gamma_0 \supset \Gamma(N) \ni \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, $g(z+N) = g(z)$. This means that g has a Fourier expansion

$$g(z) = \sum_{n=-\infty}^{\infty} a_g(n) q_N^n. \tag{51}$$

Now, we say that f is meromorphic (resp. holomorphic) at the cusp s, if $a_g(n) = 0$ for $n \ll 0$ (resp. for $n < 0$).

Note. A modular form f defined by Definition 3.2 is said to be a cusp form if $b_\gamma(0) = 0$ for all $\gamma \in SL_2(\mathbb{Z})$ in condition (iii). The set of all modular (resp. cusp) forms of weight k for the group $\Gamma_0(N)$ with character χ is denoted as $M_k(N,\chi)$ (resp. $S_k(N,\chi)$). When χ is a trivial character, then the respective spaces are denoted as $M_k(N)$ and $S_k(N)$.

Facts.

(a) $M_k(N, \chi)$ is a finite dimensional $\mathbb{C}$ -vector space.

(b) One has the Petersson inner product defined as follows.

$$\langle f, g \rangle = \int_{\mathcal{F}_N} f(z)\overline{g(z)} y^{k-2} dx dy \qquad (z = x + iy), \tag{52}$$

where $f, g \in M_k(N, \chi)$ with f or g a cusp form and $\mathcal{F}_N$ is a fundamental domain for $\Gamma_0(N)$.

(c) One has the theory of Hecke operators. In this situation, the Hecke algebra is generated by the Hecke operators T_p for $p \nmid N$ and U_p for $p|N$. These operators are defined by

$$\begin{aligned} f\Big|T_p(z) &= \frac{1}{p} \sum_{0 \le b < p} f\left(\frac{z+b}{p}\right) \;+\; \chi(p)\, p^{k-1} f(pz) \quad (p \nmid N), \\ f\Big|U_p(z) &= \frac{1}{p} \sum_{0 \le b < p} f\left(\frac{z+b}{p}\right) \quad (p|N). \end{aligned} \tag{53}$$

Here, unlike before, only the operators T_p for $p \nmid N$ are hermitian with respect to the Petersson norm and so we have the weaker form of the corresponding theorem of Hecke.

Theorem 3.1. *(Hecke-Petersson) The space $S_k(N)$ has a basis of eigenforms with respect to all Hecke operators T_p for $p \nmid N$.*

Newform theory. Note that when $N_1|N_2$, $\Gamma_0(N_2) \subseteq \Gamma_0(N_1)$ and hence $S_k(N_1) \subseteq S_k(N_2)$. In fact, $S_k \subseteq S_k(N)$ for all $N \geq 1$. The reason for the Hecke operator U_p $(p|N)$, defined on the space $S_k(N)$, not being hermitian with respect to the Petersson norm is because of this duplication of forms in higher levels. In order to find a satisfactory theory as in the case of modular forms with respect to the full modular group $SL_2(\mathbb{Z})$, A. O. L. Atkin and J. Lehner [2] defined a certain subspace of $S_k(N)$ which has the required nice properties. More precisely, they defined the subspace containing all the duplicating forms which come from lower levels as

$$S_k^{old}(N) := \left\{ f(dz) \Big| f \in S_k(r), rd|N, r \neq N \right\} \tag{54}$$

and defined the space $S_k^{new}(N)$ to be the orthogonal complement of $S_k^{old}(N)$ in $S_k(N)$ with respect to the Petersson scalar product. This space $S_k^{new}(N)$ has all the required nice properties.

Theorem 3.2. *(Atkin-Lehner)*

(a) *The space $S_k^{new}(N)$ has a basis of simultaneous eigenforms with respect to all Hecke operators.*

(b) *Let* $f \in S_k^{new}(N)$ *be a non-zero Hecke eigenform. Then,* $a_f(1) \neq 0$*. So, one can find a basis of normalised Hecke eigenforms. These are called* **newforms** *of level* N*.*

(c) *Let* $f \in S_k^{new}(N_1)$ *and* $g \in S_k^{new}(N_2)$ *be two newforms having the same eigenvalues for almost all Hecke operators, then* $N_1 = N_2$ *and* $f = g$*.*

(d) *Let* $f \in S_k^{new}(N)$ *be a newform. Then* $a_f(n)$ *(the* n*-th Fourier coefficient of* f*) is a multiplicative function. Further, it satisfies the following property.*

$$\begin{aligned} a_f(np) &= a_f(p)a_f(n) \quad \text{if } p|N, n \geq 1 \\ a_f(np) + p^{k-1}a_f(n/p) &= a_f(p)a_f(n) \quad \text{if } p \nmid N, n \geq 1. \end{aligned} \tag{55}$$

(e) *Let* $f \in S_k^{new}(N)$ *be a newform. Then the corresponding* L*-function* $L_f(s)$ *has an Euler product expansion (for* $\mathrm{Re}(s) > \frac{k}{2} + 1$*):*

$$L_f(s) = \prod_{p|N} \left(1 - a_f(p)p^{-s}\right)^{-1} \prod_{p \nmid N} \left(1 - a_f(p)p^{-s} + p^{k-1-2s}\right)^{-1}. \tag{56}$$

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