BMTSC 2022-2023 SOLUTIONS

Section 1

P1. Let the sub-segment lengths along CB be a, b, c, d respectively and those along CD be e, f, g, h respectively.

Perimeter of rectangle ABCD $= 2(a + b + c + d) + 2(e + f + g + h)$

$= 2(a + e) + 2(b + f) + 2(c + g) + 2(d + h)$

$= 48 + 24 + 16 + 32 =$ 120.

P2. $m \times n = 36 = 2^2 \times 3^2$; m and n are natural numbers.

Number of ordered pairs (m, n) satisfying given conditions = Number of positive factors of 36

$= (2 + 1)(2 + 1) =$ 9.

The actual ordered pairs are: (1, 36), (2, 18), (3, 12), (4, 9), (6, 6), (9, 4), (12, 3), (18, 2), (36, 1).

P3. The names of the 3 persons are Mr. Red, Mr. Blue and Mr. White.

From the statement made by Mr. Blue, each person's name is different from the colour of the shirt he is wearing.

Hence the man wearing the white shirt cannot be Mr. White; he is either Mr. Red or Mr. Blue. ... (1a)

Similarly, the colour of the shirt worn by Mr. Blue cannot be blue; it must be either red or white. ... (1b)

Since the man wearing the white shirt addresses another person as Mr. Blue, clearly:

(a) the man wearing the white shirt cannot be Mr. Blue; ... (2a)
(b) Mr. Blue is not wearing the white shirt. ... (2b)

From (1a) and (2a):

The man wearing the white shirt is Mr. Red ➔ Mr. Red is wearing a white shirt.

From (1b) and (2b):

Mr. Blue is wearing a red shirt.

Therefore: Mr. White is wearing a blue shirt.

P4. Let z = sqrt(9 × sqrt(9 × sqrt(9 × . . .

Therefore, we can write:

$z^2 = 9z$ ➔ $z(z - 9) = 0$ ➔ $z = 9$ or $z = 0$

Clearly, z cannot be 0. Hence we must have $z = 9$.

Hence required value $= \sqrt{4z}$ $= \sqrt{4 \times 9}$ $= \sqrt{36}$ $= 6$.

Alternatively

9 = sqrt (81) = sqrt (9 * 9) = sqrt (9 * sqrt (81)) = sqrt (9 * sqrt(9 * 9)) = sqrt (9 * sqrt(9 * sqrt(81)))

Thus sqrt(4 * 9) = sqrt (4 * sqrt(9 * sqrt(9 * sqrt(9 ….)

= 2 * 3 = 6

P5. Let the 3 digits in the original number be a, b and c – where a is in the hundreds place and c is in the units place.

The product of this 3 digit number and the number obtained by reversing its digits is given to be 83187.

Since the units place digit 7 in the product is odd, clearly neither a nor c can be even.

Hence both a and c must be odd i.e. from {1, 3, 5, 7, 9}.

We note that neither a nor c can be 5, otherwise the units place digit in the product would have been 5 – which is not the case.

Thus both a and c must be from {1, 3, 7, 9}.

We can rule out $a = c$, since none of 1×1, 3×3, 7×7, 9×9 give a 7 in the units place.

Of the remaining 6 combinations of 2 digits from {1, 3, 7, 9}, only 1×7 and 3×9 give a 7 in the units place.

The combination of a being 3 and c being 9 (or vice versa) is ruled out, since otherwise the product of the number and the number obtained by reversing the digit would necessarily be a 6 digit number exceeding 2,70,000 (300×900)– which is not the case.

Thus we must have $a = 1$ and $c = 7$ or vice versa.

Hence $83187 = 1b7 \times 7b1$.

If b is 2 or higher, then the product would at least be 84,000 (120×700)– which is not the case.

Hence b has to be either 0 or 1.

If b is 0, the product cannot possibly exceed 77,000.

Hence the only possibility left is $b = 1$.

We need to verify though if 117×711 actually tallies with the given product.

$117 \times 711 = 71{,}100 + 7{,}110 + 4{,}977 = 83{,}187$.

Hence the required number is either 117 or 711.

Comment: *The problem specifically requires us to find **all** 3 digit numbers that satisfy the given conditions. So even if a student obtains the solution purely by trial and error, the student still needs to demonstrate that there are no other 3 digit numbers as solutions.*

Alternative solution (by prime factorization):

Denote the desired 3 digit number as A and the number obtained by reversing A's digits as B.

Given: A × B = 83187.

Note that the sum of digits of 83187 is 27 i.e. a multiple of 9, which means that 9 divides 83187.

Hence A × B = 9 × 9243.

Again, the sum of digits of 9243 is 18 i.e. a multiple of 9, which means that 9 divides 9243.

Hence A × B = 9 × 9 × 1027.

Now we know that 1001 = 7 × 11 × 13 and 13 divides 26. Hence 1001 + 26 i.e. 1027 is divisible by 13.

Hence A × B = 9 × 9 × 13 × 79. Thus we obtain the prime factorization of the product 83187.

Note that 1027 = 13 × 79 is a 4 digit number. Hence both the prime factors 13 and 79 cannot be in one single number (A or B), since they are 3 digit numbers.

In other words, 13 has to be a factor of one of the numbers (say A), while 79 is a factor of the other number B. In short, A = 13m and B = 79n, where m × n = 9 x 9 = 81.

So we just need to split (i.e. factorise) 81 appropriately between A and B.

If we split 81 between A and B as 1 and 81 (or vice versa), then one of the numbers becomes divisible by 3, while the other does not. This is not possible, because the two numbers are just reverse of one another and hence the sum of digits of both must be the same.

If we split 81 between A and B as 3 and 27 (or vice versa), then one of the numbers becomes divisible by 9, while the other does not. Again, this is not possible, because the two numbers are just reverse of one another.

So the only possible split left is m = n = 9, where upon we obtain A = 13 × 9 = 711 and B = 79 × 9 = 711.

As it so happens, this does satisfy the requirement that A and B are reverse of one another.

Therefore the only such 3 digit numbers are 117 and 711.

P6. In the right-angled triangle ABC (with A as the right angle), with point D on AB, we are given:

AD = 6 AC = 20 DB + BC = AD + AC

Let the length of DB be *d*.

Hence d + BC = 6 + 20.

Therefore, BC = 26 – d. Moreover AB = AD + DB = 6 + d.

Using the Pythagoras theorem, we can write

$AB^2 + AC^2 = BC^2$ $\Rightarrow$ $(6 + d)^2 + 20^2 = (26 - d)^2$

$\Rightarrow$ $36 + 12d + 400 = 676 - 52d$

$\Rightarrow$ $64d = 240$

$\Rightarrow$ $d = 15/4 = 3.75$

Hence length of DB = 3.75 units.

P7. The only point at which all the four runners can meet simultaneously is the point of intersection of all four circular tracks i.e. the starting point.

The length of each track is 2 km.

Therefore the four runners with respective speeds 4, 6, 8 and 10 kmph will cover 1 lap of their allocated track in ½, ⅓, ¼, ⅕ hour respectively i.e. in 30, 20, 15, and 12 minutes respectively.

The smallest natural number that is divisible by each of 30, 20, 15, and 12 is 60.

Hence at the end of 60 minutes i.e. 1 hour, all 4 runners would meet once again at the starting point for the first time – from the time they started their runs.

Therefore, they will meet simultaneously 3 times at the end of 3 hours – when they all stop their runs.

Thus each of them runs for 3 hours.

P8. LCM of two natural numbers = 640; their product = 10240.

We know that: GCD of two natural numbers × their LCM = their product.

Hence, GCD of the required numbers × 640 = 10240

$\Rightarrow$ GCD of the required numbers = 16.

Hence the two numbers are of the form 16a and 16b, where a and b are relatively prime natural numbers.

Hence we must have: $16a \times 16b = 10240$ $\Rightarrow$ $a \times b = 40 = 2^3 \times 5$, a and b are relatively prime.

The only ways to write 40 as the product of two relatively prime natural numbers are:

1 × 40 or 5 × 8.

Correspondingly the required pair of natural numbers are: 16, 640 or 80, 128.

Section 2

P1.

The required 7 digit number has to use the digits 1, 3, 5, 7 and 9 at least once and has to be divisible by both 5 and 9.

Let the two remaining digits in the number be x and y.

Sum of digits of the number = $1 + 3 + 5 + 7 + 9 + x + y = 25 + x + y$.

The minimum sum of digits achievable is 25 + 1 + 1 = 27.

The maximum sum of digits achievable is 25 + 9 + 9 = 43.

Since the number has to be divisible by 9, the sum of its digits must be a multiple of 9 in the closed interval [27, 43].

The only multiples of 9 in this range are 27 and 36.

Hence either $x + y = 2$ or $x + y = 11$.

Since both x and y are odd digits, their sum has to be even and hence we cannot have $x + y = 11$.

Thus we must have $x + y = 2$, with $x = y = 1$.

Hence the 7 digits are: 1, 3, 5, 7, 9, 1 and 1.

Since the number has to be divisible by 5, the units place digit has to be 5. The remaining 6 digits viz. 1, 3, 7, 9, 1 and 1 simply need to be arranged in descending order to maximize the desired number.

Hence the maximum such 7 digit number is: 97,31,115.

P2. Let the 5 numbers, whose iterative average is to be computed, be a, b, c, d, e – arranged in that order itself.

Hence iterative average of a, b, c, d, e $= 0.5(0.5\{0.5[0.5(a + b) + c] + d\} + e)$

$= 0.0625a + 0.0625b + 0.125c + 0.25d + 0.5e$

As can be seen, in this iterative average of a, b, c, d, e in that order, we have:

Weight of e > weight of d > weight of c > weight of b = weight of a

Thus, for a given set of numbers, the iterative average is maximal if the numbers are arranged in ascending order – while it is minimal if the numbers are arranged in descending order.

The given numbers in this problem are 1, 2, 3, 4 and 5.

Hence maximum iterative average for these numbers is:

$$0.0625 \times 1 + 0.0625 \times 2 + 0.125 \times 3 + 0.25 \times 4 + 0.5 \times 5$$

The minimum iterative average of these numbers is:

$$0.0625 \times 5 + 0.0625 \times 4 + 0.125 \times 3 + 0.25 \times 2 + 0.5 \times 1$$

Therefore difference between the maximum and minimum iterative averages is:

$$0.5 \times (5 - 1) + 0.25 \times (4 - 2) + 0.0625 \times (2 - 4) + 0.0625 \times (1 - 5)$$

$$= 2 + 0.5 - 0.125 - 0.25 \quad = 2 + 0.5 - 0.375 \quad = 2 + 0.125 \quad = 2.125$$

Thus the difference between the maximum and minimum iterative averages of the numbers 1, 2, 3, 4 and 5 is: 2.125 = 17/8.

P3. Given QT = ST = RT.

Hence $\triangle$TRS is isosceles with TR = TS ➔ $\angle$TRS = $\angle$TSR ... (1)

Also TSQ is isosceles with TS = TQ -> angle TSQ = angle TQS ... (2)

Given PQ = QS = SR.

Hence $\triangle$QRS is isosceles with SR = QS ➔ $\angle$QRS = $\angle$RQS ... (3)

But angles TRS and QRS are same and angle TQS and RQS are same.

So from 1, 2 and 3,

The sum of TRS, RST, TSQ and SQT is 180 as they are interior angles in triangle RSQ

So each is 45 degrees

On parallel lines ST and PQ and transversal SQ, angles QST and SQP are alternate angles. So they are equal. Therefore angle SQP = 45 degrees.

Therefore angle TQP = 90 degrees

On parallel lines ST and PQ and transversal TQ, angles RTS and TQP are corresponding angles. So they are equal.

Angle RTS = 90 degrees

In isosceles triangle SQP, PQ = SQ. So angles QSP = QPS = x and SQP = 45

So x + x + 45 = 180

So QPS = x = 67.5 degrees

P4. Number of squares in 1st ring (3 × 3 big square) = $3^2 - 1^2 = (3 - 1)(3 + 1) = 2 \times 4 = 8$.

Number of squares in 2nd ring (5×5 big square) = $5^2 - 3^2 = (5 - 3)(5 + 3) = 2 \times 8 = 16$.

Number of squares in 3rd ring (7×7 big square) = $7^2 - 5^2 = (7 - 5)(7 + 5) = 2 \times 12 = 24$.

Thus, number of squares in n^{th} ring ($2n + 1 \times 2n + 1$ big square) = $(2n + 1)^2 - (2n - 1)^2 = 8n$.

Therefore, number of squares in 25th ring (51×51 big square) = $8 \times 25 = 200$.

The ring colour sequence is as follows:

1st ring: Yellow;	2nd ring: Green;	3rd ring: Red;
4th ring: Yellow;	5th ring: Green;	6th ring: Red;
. . .		
22nd ring: Yellow;	23rd ring: Green;	24th ring: Red;
25th ring: Yellow.		

Hence green unit squares are located only in the following rings:

2nd, 5th, 8th, . . . , 20th, 23rd

Therefore number of green unit squares = Number of squares in 2nd ring +
Number of squares in 5th ring +
Number of squares in 8th ring +
. . . +
Number of squares in 23rd ring

$= 8 \times 2 + 8 \times 5 + 8 \times 8 + . . . + 8 \times 23$

$= 8 \times (2 + 5 + . . . + 20 + 23)$

$= 8 \times (4 \times 25)$

$= 800$.

Thus, there are 800 green coloured unit squares in the 51×51 big square.