Why logic works

Lecture in honour of S S Abhyankar, 22 July 2020

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- This is the job of logic : it guarantees the truth of any inference made from any true hypothesis, provided the argument is logically valid.
- It has been fairly successful in this job. Our everyday life depends on it. Engineers, doctors, lawyers rely on it for life-and-death decisions. All academic subjects, especially, the edifice of mathematics, is a testimony that logic works.

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- What are the **alternatives** to standard logic?
- When are the conditions needed by logic not fulfilled? How can we understand such things?

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- A remarkable feature of all this is that logical validity of an argument depends only on its **form**, not on what it refers to.

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- The checking involves only **for** loops, and does not need **until** loops, in computer programming terms.

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- A **compound sentence** is formed from simple sentences through **logical connectives** and **quantifiers**, and using brackets.

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- Both languages have the binary relation $=$ ('equal to').

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- Exercise: Translate into L_A the sentence 'x is a prime'.

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- Baba vakyam pramanam. (Whatever Baba says is true.)

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- The concept of logical truth extends to open sentences.

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Theorem (Church and Turing, 1930's) Logical truth is undecidable.

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Clearly, if $\{P_1, P_2, \ldots\} \vdash A_n$ then $\{P_1, P_2, \ldots\} \models A_n$.
Important question Is the converse true?

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- Logical axioms: Three propositional axiom schema.
  (1) $A \Rightarrow (B \Rightarrow A)$
  (2) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)$
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- Three axiom (schema) of equality. (E1) $(\forall x)(x = x)$.
  (E2) Replacing a term by an equal term inside a function gives equal values.
  (E3) Replacing a term by an equal term inside a relation gives a new statement which is implied by the old statement.

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- This does not contradict the undecidability of logical truth!

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Example: The theory PA (Peano Arithmetic) has L_A as its language, the Peano Axioms as its axioms, and $D = \{0, 1, 2, 3, \ldots\}$ with the usual interpretations for $=$, 0, 1, S, $+$, $\times$.

Peano axioms

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- Robinson axioms.
  (1) $(\forall x)(Sx \neq 0)$
  (2) $(\forall x)(\forall y)(Sx = Sy \Rightarrow x = y)$
  (3) $(\forall x)((x \neq 0) \Rightarrow (\exists y)(Sy = x))$
  (4) $(\forall x)(x + 0 = x)$
  (5) $(\forall x)(\forall y)(x + Sy = S(x + y))$
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- Induction axiom schema. For each A we have an axiom:
  $(A(0) \wedge (\forall x)(A(x) \Rightarrow A(Sx))) \Rightarrow (\forall x)A(x)$.
  If A has other free variables besides x, then universally quantify the above formula over them.

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- If the language L has a (defined) predicate which says 'x is a natural number', and the standard arithmetical symbols $=$, 0, 1, S, $+$, $\times$ are available in L, then there is a purely arithmetical predicate $Thm(n)$ in L, which says that n is a natural number which codes a formal theorem in the theory T.

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- T is **consistent** if there is no sentence A such that both A and $\neg A$ are theorems of T. Equivalently, $\neg(0 = 0)$ should not be a theorem of T. If g denotes the Gödel number of $\neg(0 = 0)$, then T is consistent if and only if $\neg Thm(g)$ is a true statement of arithmetic.

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- Tarski's theorem on formal undefinability of truth within L.

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- Today, training in the form of advanced PhD level courses in logic is mostly confined to the computer sciences departments of universities.
- Machine learning has to some extent bypassed the need of a clear formal understanding of a process by us, before we can ask a computer to practically carry it out with a high chance of success.

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- First, the physicists make various mathematical models. In current era, we expect these to be in terms of definitions made via standard Bourbaki-style mathematics. While a physicist is usually quite informal about it, the ideal expectation is to have clear unambiguous definitions of a model. For example, you may define **a spacetime** to mean a pair (M, g) where M is any 4-dimensional smooth manifold with a semi-riemannian metric g of signature $(-1, 1, 1, 1)$.

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- Logical arguments within the model are carried out as mathematical arguments, without direct reference to foundations of mathematics and formal logic.
- The philosophically more difficult part is the going back and forth between experimental evidence about 'physical reality' and the mathematical models. There is no formal theory for it.

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However, the real world – where we must use natural languages and good mathematical models are hard to come by – seems to be much more complicated!

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- The arguments made by lawyers are often of the kind called 'lawyer-like arguments'. I will not say anything more about them.
- Not unsurprisingly, formal logic – beyond some basic clarity in the use of expressions such as 'if', 'only if', 'and' 'or' 'not' 'for all' etc. – is quite useless in most disciplines away from mathematics or CS. A basic higher secondary education in mathematics can give an adequate grasp of this.

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There is life before and beyond logic –
we necessarily go on as integral parts of the universe!