ARITHMETIC MEAN IN ANCIENT INDIA AND EUROPE

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The importance of the Arithmetic Mean was articulated in 1809 by C.F. Gauss, one of the pioneers of mathematical statistics, as follows: "it has been customary certainly to regard as an axiom the hypothesis that if any quantity has been determined by several direct observations, made under the same circumstances and with equal care, the arithmetic mean of the observed values affords the most probable value, if not rigorously, yet very nearly at least, so that it is always safe to adhere to it."

Like the decimal system, the Arithmetic Mean now appears as a natural concept to us who have grown up with it. But the history of the concept shows that its introduction must have involved conceptual subtleties. S.M. Stigler, a distinguished statistician of our time, considers the Arithmetic Mean as the first of the "five ideas that changed statistics and continue to change the way we think about the world".

The Arithmetic Mean has been variously perceived: as an exact mathematical concept, as an applied tool for combining measurements in experimental sciences, as a measure of the central tendency in statistical data, etc. Any measure of the central tendency seeks to identify that [central] value of the distribution which can be taken to be its best representative. In this talk, we shall discuss the emergence of some of the avatars of the Arithmetic Mean at different time points, in different contexts, in different mathematics cultures.

The word "mean" is derived from Old French *meien* meaning "middle" or "centre". And indeed the Arithmetic Mean captures the concept of the "centre". In fact, the first recorded definition of Arithmetic Mean is as a geometry concept in ancient Greece (c. 500 BCE): a number equidistant from two given numbers, i.e., as the "middle" point of two numbers. But it does not appear to have been envisaged as a statistics concept (average or best representative).

The general Arithmetic Mean $\overline{x} = \frac{x_1+x_2+\cdots+x_n}{n}$ and its statistical applications have not been found in European treatises prior to the sixteenth century CE. The formulation $\overline{x} = \frac{x_1+x_2+\cdots+x_n}{n}$ (whether expressed through symbols or words) is something intrinsically algebraic and, for effective application, requires good algorithms for performing all the elementary arithmetic operations. The appearance of the arithmetic mean in Europe coincides with her adoption of the decimal system and arithmetic methods based on the decimal system (which are, in essence, of Indian origin) and with the emergence of algebra.

A basic idea at the heart of the Arithmetic Mean as a statistical estimate is to combine observations — to replace several numbers by a single number. Thinkers on statistics like Stigler feel that this idea is counterintuitive: for it seeks, paradoxically, to gain information about the data by discarding information, namely, the individuality of the observations. This could also explain the delay among European scientists in developing a precise method (like Arithmetic Mean) for obtaining a best estimate on the basis of several observations. They relied on their judgements to select a particular observation thought to be the best rather than combine (and thereby spoil) it with other observations.

The technique of repeating and combining measurements on the same quantity can be seen in the work of Tycho Brahe and a few other astronomers towards the end of the sixteenth century, but the method for combining the repeated observations into a single number is not stated explicitly. Historians of statistics have usually identified the earliest unambiguous statistical use of the Arithmetic Mean in a work of the English astronomer Henry Gellibrand (1635). In his Presidential address at the American Statistical Association in 1971, Churchill Eisenhart stated, "I fully expected that I would find some good examples of mean-taking in ancient astronomy; and, perhaps, also in ancient physics. I have not found any. And I now believe that no such examples will be found in ancient science."

But, as in the case of numerous other scientific concepts, the above account of history completely overlooks the clear, precise and abundant use of the Arithmetic Mean in the treatises of ancient Indian mathematicians like Brahmagupta (628 CE), Śrīdharācārya (c. 750 CE), Mahāvīrācārya (850 CE), Pṛthūdakasvāmī (864 CE), Bhāskarācārya (1150 CE) and others. In fact, the Indian mathematicians define and apply the more general and sophisticated concept of the *weighted* Arithmetic Mean. An intuitive awareness of the law of large numbers for Arithmetic Mean also comes out in a commentary by the mathematician Gaṇeṣa (1545 CE).

The chapters in Indian arithmetic called *khāta-vyavahāra* (mathematical processes pertaining to excavations) describe how to compute the [average] depth, width or length of an irregular-shaped pool of water and thereby estimate its volume. It is in these chapters that the Arithmetic Mean is defined and used in the statistical sense — as the best representative value for a set of observations. Brahmagupta is one of the earliest Indian mathematicians in whose text the concept of [weighted] Arithmetic Mean occurs explicitly in this sense. He uses it to represent the depth of a ditch, when the depth is different in different portions of the ditch. The ancient Indian terms for Arithmetic Mean, like *sama-rajju* (mean measure of a line segment) or *samamiti* (mean measure) which use the word *sama* (equal, equable, same, common, mean) confirm that the Arithmetic Mean was perceived by ancient Indian scholars as the "common" or "equalizing" value which would be the appropriate representative measure for various observed measurements. The treatment of the excavation problems through the Arithmetic Mean was possibly one of the factors which shaped the gradual development of calculus in India.

The weighted arithmetic mean also appears in ancient Indian arithmetic treatises as an exact mathematical formula in chapters on *miśraka-vyavahāra* (computations pertaining to mixtures) which address the problem of computing the proportion of pure gold in an alloy formed by blending of several pieces of gold of different weights and purities.

Ancient India had a strong tradition in computational mathematics and algebra which provided a conducive mathematical environment for the emergence of the statistical arithmetic mean. The conceptualization of the Arithmetic Mean involves the idea of *combining* several numbers into a single number. Ancient Indian mathematics is replete with various ideas of "combination". One is reminded of the profound *bhāvanā* of Brahmagupta — a law of composition which *combines* two solutions of a certain quadratic equation in three variables to produce another solution of the equation, a principle with momentous consequences in mathematics.