Tutorial Problems
Complied by
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1. If n is a positive integer, show that $|Imz^n| \leq n\,|Im\ z|\,|z|^{n-1}$.
2. For distinct complex numbers z and a, prove that $\left|\frac{z^n-a^n}{z-a}\right| \leq \frac{|z|^n-|a|^n}{|z|-|a|}$.
3. Prove $\left|\frac{a-b}{1-\overline{a}b}\right| < 1$, if $|a| < 1$ and $|b| < 1$.
4. Evaluate the expression for $|z| < 1$:

$$(1+z+z^2+...z^9)(1+z^{10}+...z^{90})(1+z^{100}+...+z^{900}).....$$

5. Let $z_1, z_2 \in \mathbb{C}$ such that $|z_1| > |z_2|$. Show that for all $n \geq 2$,

$$n\left|\frac{z_2}{z_1}\right|^{n-1} < \frac{|z_1|}{|z_1|-|z_2|}$$

6. Show that all circles passing through a and $\frac{1}{\overline{a}}$ cut $|z| = 1$ at right angles. (Give a geometric proof)
7. Prove that $|z-1| \leq (|z|-1) + |z|\,|argz|$. Use Geometric arguments.
8. **Circles of Appolonius** Discuss the locus of the points given by $\left|\frac{z-1}{z+1}\right| = r$, $r \geq 0$. What happens for $r = 0, 1$?
9. Show that z and z' correspond to diametrically opposite points of Riemann Sphere if and only if $z\overline{z'} = -1$.
10. For what values of z is $\sum\left(\frac{z}{1+z}\right)^n$ convergent?
11. Let $a_0, a_1, ...$ be defined by $1 - x^2 + x^4 - x^6 + ... = \sum_{n\geq 0} a_n(x-3)^n, x \in (0,1)$. Evaluate $limsup\,|a_n|^{\frac{1}{n}}$.
12. If $f(z) = \sum_{n\geq 0} a_n z^n$ has radius of convergence $R > 0$, show that $h(z) = \sum_{n\geq 0} \frac{a_n z^n}{n!}$ is an entire function.
13. Show that there is no power series $f(z) = \sum c_n z^n$ such that $f(z) = 1$ for $z = \frac{1}{2}, \frac{1}{3}, ...$ and $f'(0) \neq 0$.

14. Let $P(z)$ be a polynomial. Show that if all zeros of $P(z)$ lie on one side of a line L then all zeros of $P'(z)$ also lie on same side of L. Deduce that (**Gauss-Lucas Theorem**) the smallest convex polygon that contains zeros of $P(z)$ also contains all zeros of $P'(z)$.

15. Let $f(z)$ be analytic in a region D, and $0 \in D$. Suppose that $f'(z) = f(z),\ \forall z \in D$. Show that $f(0) = 1$ if and only if $f(z) = exp(z)$.

16. If f is analytic in a region G and that every point, either $f = 0$ or $f' = 0$, show that f is a constant.

17. Discuss : $Lt_{z\to\infty} e^z$.

18. Suppose $f : G \to \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is a constant.

19. Let G be a region and define $G^* = \{z : \overline{z} \in G\}$. If $f : G \to \mathbb{C}$ is analytic prove that $f^* : G^* \to \mathbb{C}$, defined by $f^*(z) = \overline{f(\overline{z})}$, is also analytic.

20. Suppose that $f : G \to \mathbb{C}$ is a branch of the logarithm and that n is an integer. Prove that $z^n = exp(nf(z))$ for all z in G.

21. Show that the real part of the function $f(z) = z^{\frac{1}{2}}$ is always positive iff $f(1) = 1$.

22. Give the principal branch of $\sqrt{1-z}$.

23. For $r > 0$ let $A = \left\{\alpha : \alpha = exp(\frac{1}{z}), 0 < |z| < r\right\}$; Describe A geometricaly, or give an alternative and 'better' description of it.

24. Prove that there is no branch of logarithm defined on $G = \mathbb{C} - \{0\}$.

25. For the function, $f(z) = \frac{\overline{z}^2}{z}, z \neq 0$ and $f(0) = 0$, Show that the Cauchy Riemann equations are satisfied at $(0, 0)$ but the function is not differentiable at $0 + i0$.

26. Prove : $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \overline{z} \partial z}$.

27. Let f be an analytic function, $f = u + iv$. Prove that $\frac{\partial f}{\partial \overline{z}} = 0$.

28. If f is analytic, prove that $\overline{f}$ is independent of z.

29. Let $f(z)$ be an analytic function. Then prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2.$$

30. Show that there is no analytic function $f = u+iv$ with $u(x,y) = x^2+y^2$.

31. Sketch the level curves of real and imaginary parts of $f(z) = z^2$. Are they orthogonal to each other? Can you conclude something from this?

32. Let $f(z) = z^2; g(z) = 1/z$ and a, b, c, d be real numbers. Find the images under f, g of the following susbets of the complex plane.
(i) the real axis, (ii) the imaginary axis,
(iii) the lines $x = a$; (iv) the lines $y = b$;
(v) the first quadrant; (vi) the upper half plane;
(vii) the second quadrant; (viii) first quadrant intersected with unit disc;
(ix) the hyperbolae $x^2 - y^2 = c$; (x) the hyperbolae $2xy = d$.
(xi) the circles $x^2 + y^2 = b^2$; (xii) the half-plane $y > 0$.
(xiii) the infinite strip $0 < y < 1$.

33. Find the image of lines $x = a$ and $y = b$ under the map $f(z) = sinz$ where a, b are real numbers.

34. Let $f(z) = e^z$. Find the images of lines $x = a$, lines $y = b$, Strip $-\pi < Imz < \pi$, strip $a < Rez < b$ where a, b are real numbers.

35. Let $f(z) = logz$, the principal branch of logarithm. Find the images of
(i) the sector, $0 < Argz < \alpha$, where $\alpha < \frac{\pi}{2}$ and
(ii)$\mathbb{C} - \{z : Rez \leq 0\}$.

36. Find a Möbius transformation mapping the set A onto the set B where A, B are given by
(i) $A = \{z : Rez \leq 1\}$; $B\,\{z : |z-2| \geq 2\}$.
(ii) $A = \{z : Rez \geq 0, Imz \geq 0, |z| \geq 1\}$ and $B = \{z \ : \ |z| < 1\}$
(iii) A= Upper Half plane; B= interior of unit disc.
(iv)A= Right Half plane; B= interior of unit disc.
(v) A is the strip $-1 < Rez < 1$; B= the interior of unit disc.
(vi) A is the $\frac{-\pi}{2} < Imz < \frac{\pi}{2}$; B = the interior of unit disc.

37. Find the image of $\{z : Rez < 0, |Imz| < \pi\}$ under the exponential function.

38. Discuss the mapping properties of the function z^n and $z^{\frac{1}{n}}$ for $n \geq 2$.

39. Evaluate the Cross ratios : $(0, 1, i, -1)$ and $(4 + i, 1, 0, \infty)$.

40. If $Tz = \frac{az+b}{cz+d}$ find z_2, z_3, z_4 in terms of a, b, c, d such that $T(z) = (z, z_2, z_3, z_4)$.

41. Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is a constant.

42. Does there exist an analytic function f such that $f(\frac{1}{n}) = f(\frac{-1}{n}) = \frac{1}{n^2}$.

43. Does there exist an analytic function g such that $g(\frac{1}{n}) = g(\frac{-1}{n}) = \frac{1}{n^3}$.

44. Let f be an entire function such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$. Is f constant? Justify.

45. Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all a in G. Show that either $f \equiv 0$ or $g \equiv 0$.

46. Let $U : \mathbb{C} \to R$ be a harmonic function such that $U(z) \geq 0$ for all z in $\mathbb{C}$; prove that U is a constant.

47. Show that if f and g are analytic functions on a region G such that $\overline{f}g$ is analytic then either f is a constant or $g \equiv 0$.

48. For $|z| < 1$ define $f(z) = exp\left\{-ilog[i(\frac{z+1}{1-z})]\right\}$. Find image of the open unit disc under f.

49. Let f be an entire function and suppose there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

50. Find all entire functions f such that $f(x) = e^x$ for $x \in R$.

51. Let G be a region and suppose that $f : G \to \mathbb{C}$ is analytic and $z \in G$ such that $|f(a)| \leq |f(z)|$ for all z in G. Show that $f(a) = 0$ or f is constant.

52. Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| > R\}$. If $\gamma(t) = Re^{it}, 0 \leq t \leq 2\pi$, show that $\int_\gamma \frac{p'(z)}{p(z)} dz = 2\pi i n$.

53. Let f be analytic on $D = B(0;1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show that $|f'(0)| \leq 1$.

54. Let G be region and suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is analytic.

55. Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$ then f is an entire function.

56. Prove that an entire function has a removable singularity at infinity iff it is a constnat.

57. Prove that an entire function has a pole at infinity of order m iff it is a polynomial of degree m,

58. Characterize those rational functions which have a removable singularity at infinity

59. Characterize those rational functions which have a pole or order m at infinity.

60. If $f : G \to \mathbb{C}$ is analytic except for poles show that the poles of f cannot have a limit point in G.

61. Let $\lambda \in \mathbb{C}$ and show that

$$exp\left\{\tfrac{1}{2}\lambda(z + \tfrac{1}{z})\right\} = a_0 + \textstyle\sum_{n\geq 1} a_n(z^n + \frac{1}{z^n})$$

for $0 < |z| < \infty$, where for $n \geq 0$,

$$a_n = \tfrac{1}{\pi} \textstyle\int_0^\pi e^{\lambda \cos t} \cos nt dt.$$

62. Determine the regions in which the functions $f(z) = sin(\frac{1}{z})^{-1}$ and $g(z) = \int_0^1 (t-z)^{-1}dt$ are analytic. Do they have any isolated singularities? Do they have any singularities that are not isolated?

63. Suppose that f has a simple pole at $z = a$ and g is analytic in an open set containing a. Show that $Res(fg; a) = g(a)Res(f; a)$.

64. Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z = x + iy : x > 0\}$. Show that this solution must be real.

65. Supoose f is analytic in $\overline{B}(0; 1)$ and satisfied $|f(z)| < 1$ for $|z| = 1$. Find the numebr of soultins (counting multiplicities) of the equation $f(z) = z^n$ where n is an integer larger than or equal to 1.

66. Let f be meromorphic on G; show that neither the poles nor the zeros of f have a limit point in G.

67. Find number of zeros of the polynomial $z^7 - 4z^3 + z + 1$ inside the open unit disc.

68. Find number of zeros of $z^8 - z^7 - 4z^2 - 1$ in the annulus $1 < |z| < 2$.

69. Suppose $|f(z)| \leq 1$ for $|z| < 1$ and f is analytic. By considering $g : D \to D$ defined by $g(z) = \frac{f(z)-a}{1-\overline{a}f(z)}$ where $a = f(0)$, prove that

$$\frac{|f(0)|-|z|}{1+|f(0)||z|} \leq |f(z)| \leq \frac{|f(0)|+|z|}{1-|f(0)||z|}$$

70. Does there exist an analytic function $f : D \to D$ with $f(\frac{1}{2}) = \frac{3}{4}$ and $f'(\frac{1}{2}) = \frac{2}{3}$, where D is the open unit disc.

71. Suppose $f : D \to D$ satisfies $Re\ f(z) \geq 0$ for all z in D and suppose that f is analytic.
(a) Show that $Re\ f(z) > 0$ for all z in D.
(b) By using an appropriate Möbius transformation, prove that if $f(0) = 1$ then,

$$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}$$

72. Let f be analytic in $D = \{z : |z| < 1\}$ and suppose that $|f(z)| \leq M$ for all z in D. If $f(z_k) = 0$ for $1 \leq k \leq n$ show that

$$|f(z)| \leq M \prod_{k=1}^{k=n} \left|\frac{z-z_k}{1-\overline{z_k}z}\right|.$$

73. Suppose f is analytic in some region containing $\overline{B(0;1)}$ and $|f(z)| = 1$ where $|z| = 1$. Find a formula for f. (Hint : First consider the case where f has no zeros in $B(0;1)$.