## § 4 Projective Real Nullstellensatz

The results proved in this section are based on the personal discussions of second author (Dilip P. Patil) with Professor Uwe Storch, Ruhr-Universität Bochum, Germany and his lecture on 23 January 2003, on the occasion of 141-th birthday of Hilbert at the Ruhr-Universität Bochum, Germany.

The main aim of this section is to prove the Projective Real Nullstellensatz : Homogeneous polynomials $f_1,\ldots,f_r \in \mathbb{R}[T_0,\ldots,T_n]$, $r \le n$, of positive *odd* degrees in $n+1$ indeterminates have a common non-trivial zero in $\mathbb{R}^{n+1}$, or equivalently — a common zero in $n$-dimensional projective space $\mathbb{P}^n(\mathbb{R})$ over $\mathbb{R}$.

Our proof of the Projective Real Nullstellensatz is elementary and uses standard definitions and basic properties of Poincaré series, projective (krull) dimension and multiplicity of (standard) graded algebras over a field. This proof the fundamental property of the real numbers, namely : every odd degree polynomial over the field of real numbers has a real root. We say a field $K$ is a 2-*field* if every odd degree polynomial over $K$ has a root in $K$. Therefore the Projective Real Nullstellensatz can be generalized for 2-fields. As an application, we prove the well-known Borsuk-Ulam Theorem.

**4.1 Notation and Preliminaries** Let $K$ be a field and let $\mathrm{P} := A_0[T_0,\ldots,T_n]$ be the polynomial algebra in indeterminates $T_0,\ldots,T_n$ over a commutative ring $A_0$. The homogeneous polynomials of degree $m \in \mathbb{N}$ form an $A_0$-submodule $\mathrm{P}_m$ of P and $\mathrm{P} = \bigoplus_{m\in\mathbb{N}} \mathrm{P}_m$. Further, $\mathrm{P}_m\mathrm{P}_k \subseteq P_{m+k}$ for all $m,k \in \mathbb{N}$ and as an $A_0$-algebra P is generated by homogeneous elements $T_0,\ldots,T_n$ of degree 1.

**(1) Graded rings and Modules** More generally, a ring $A$ is called $\mathbb{N}$-*graded* or just *graded* if it has a direct sum decomposition $A = \bigoplus_{m\in\mathbb{N}} A_m$ as an abelian group such that $A_mA_k \subseteq A_{m+k}$ for all $m,k \in \mathbb{N}$. In particular, $A_0$ is a subring of $A$ and $A_m$ is an $A_0$-module for all $m \in \mathbb{N}$. For $m \in \mathbb{N}$, $A_m$ is called *homogeneous component* of $A$ of degree $m$ and its elements are called *homogeneous elements of degree m*.

A graded ring $A = \bigoplus_{m\in\mathbb{N}} A_m$ is called a *standard graded $A_0$-algebra* if $A$ is generated by finitely many homogeneous elements of degree 1 as an $A_0$-algebra. A standard example of the standard graded $A_0$-algebra (as seen above) is the polynomial algebra $\mathrm{P} = A_0[T_0,\ldots,T_n]$ with $\deg T_i = 1$ for all $i = 0,\ldots,n$ over a commutative ring $A_0$.

Let $A = \bigoplus_{m\in\mathbb{N}} A_m$ be a graded ring and $A_+ := \bigoplus_{m\in\mathbb{N}^+} A_m$. Obviously, $A_+$ is an ideal in $A$ called the *irrelevant ideal* of $A$. It follows that the following statements are equivalent : (i) $A$ is Noetherian. (ii) $A_0$ is Noetherian and $A_+$ is finitely generated. (iii) $A_0$ is Noetherian and $A$ is an $A_0$-algebra of finite type.

**Remark :** Note that finite type algebras over Noetherian ring are Noetherian. However, Noetherian algebras over a Noetherian ring $A_0$ are not always of finite type over $A_0$. Therefore graded rings are special in which this converse holds.

A *graded module* over the graded ring $A = \bigoplus_{m\in\mathbb{N}} A_m$ is an $A$-module $M$ with a direct sum decomposition $M = \bigoplus_{m\in\mathbb{Z}} M_m$ as an abelian group such that $A_mM_k \subseteq M_{m+k}$ for all $m \in \mathbb{N}$ and all $k \in \mathbb{Z}$. In particular, $M_m$ is an $A_0$-submodule of $M$ for every $m \in \mathbb{Z}$. For $m \in \mathbb{Z}$, $M_m$ is called *homogeneous component* of $M$ of degree $m$ and its elements are called *homogeneous elements of degree m*.

Let $M = \bigoplus_{m\in\mathbb{Z}} M_m$ and $N = \bigoplus_{m\in\mathbb{Z}} N_m$ be graded $A$-modules over the graded ring $A = \bigoplus_{m\in\mathbb{N}} A_m$. An $A$-module homomorphism $f : M \to N$ is called *homogeneous of degree r* if $f(M_m) \subseteq N_{m+r}$ for every $m \in \mathbb{Z}$.

An $A$-submodule $M'$ of the graded $A$-module $M$ is called *homogeneous* if $M'_m := \pi_m(M') = M' \cap M_m \subseteq M'$, where $\pi_m : M \to M_m$, $m \in \mathbb{Z}$ are the projections of the graded $A$-module $M$. If the $A$-submodule $M' \subseteq M$ is homogeneous, then $M' = \bigoplus_{m\in\mathbb{Z}} M'_m$ is a graded $A$-module and the canonical injective map $M' \to M$ is homogeneous of degree 0. An $A$-submodule $M'$ of the graded $A$-module $M$ is homogeneous if and only if $M'$ has a generating system consisting of homogeneous elements. Further, the residue-class module $M/M'$ has the direct sum decomposition $M/M' = \bigoplus_{m\in\mathbb{Z}} \overline{M}_m$, where $\overline{M}_m := M_m/M'_m$. Obviously, $M/M'$ with this gradation is a graded $A$-module and the canonical surjective map $M \to M/M'$ is homogeneous of degree 0.

An ideal $\mathfrak{a} \subseteq A$ is called *homogeneous* if $\mathfrak{a}$ is a homogeneous submodule of $A$.

Let $f : M \to N$ be a homogeneous homomorphism of degree $r$, then $\operatorname{Ker} f$ and $\operatorname{Im} f$ are homogeneous submodules of $M$ and $N$, respectively and the canonical 4-term sequence

$$0 \to \operatorname{Ker} f \longrightarrow M \longrightarrow N \longrightarrow \operatorname{Coker} f \to 0$$

is an exact sequence of graded $A$-modules and homogeneous homomorphisms. Further, for every $m \in \mathbb{Z}$, the sequence of $A_0$-modules

$$0 \to (\operatorname{Ker} f)_m \longrightarrow M_m \longrightarrow N_{m+r} \longrightarrow (\operatorname{Coker} f)_{m+r} \to 0$$

is exact.

**(a) Shifted graded modules** The following shift operation is very useful: For $k \in \mathbb{Z}$, a graded $A$-module $M(k)$ obtained from the graded $A$-module $M = \bigoplus_{m\in\mathbb{Z}} M_m$ with $M(k)_n := M_{k+n}$ for all $n \in \mathbb{Z}$ is called the *k-th shifted graded A-module* of $M$. In particular, we have the $k$-shifted graded $A$-module $A(k)$ of the graded $A$-module $A$. Clearly, an $A$-module homomorphism $f : M \to N$ is homogeneous of degree $r$ if and only if $f : M(-r) \to N$, or $f : M \to N(r)$ is homogeneous of degree 0.

**(b) Noetherian graded modules** We consider on the case when $A_0 = K$ is a field and $A$ is a standard graded $K$-Algebra. If $t_0,\ldots,t_n \in A_1$ generates $A$ as a $K$-algebra, i. e. $A = K[t_0,\ldots,t_n]$, then the $K$-algebra substitution homomorphism $\varepsilon : K[T_0,\ldots,T_n] \to A$ with $T_i \to t_i$, $i = 0,\ldots,n$, is homogeneous and surjective, and hence $A$ is isomorphic to the residue-class algebra $K[T_0,\ldots,T_n]/\mathfrak{A}$ of $\mathrm{P} = K[T_0,\ldots,T_n]$ modulo the homogeneous *relation ideal* $\mathfrak{A} := \operatorname{Ker}\varepsilon$. Every $A$-module is also P-module by the restriction of scalars by using $\varepsilon$. We consider graded $A$-modules $M$ which are finite over $A$, i. e. finitely generated over $A$. If $x_1,\ldots,x_r$ is a homogeneous generating system for $M$ of degrees $\delta_1,\ldots,\delta_r \in \mathbb{Z}$, then the canonical homomorphism $A(-\delta_1)\oplus\cdots\oplus A(-\delta_r) \longrightarrow M$ with $e_\rho \mapsto x_\rho$, $\rho = 1,\ldots,r$, is homogeneous (of degree 0) and surjective. The standard basis element $e_\rho \in A(-\delta_\rho)$ has the degree $\delta_\rho$. *If $A$ is a standard graded $K$-algebra and if $M$ is finite $A$-module, then $M$ is a Noetherian*[11] *A-module*, i. e. every $A$-submodule of $M$ is also a finite $A$-module. This is equivalent with the condition that in $M$ there is no infinite proper *ascending* chain $M_0 \subset M_1 \subset M_2 \subset \cdots \subseteq M$ of $A$-submodules, or also with the condition that every non-empty set of $A$-submodules of $M$ has a (at least one) maximal element (with respect to the inclusion).

We will use the following fundamental lemma on the *Lasker-Noether decomposition* [12] :

**Lemma** (Lasker-Noether decomposition) *Let $M$ be a finite graded module over the standard graded $K$-algebra $A$. Then there exists a chain of homogeneous submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$, and homogeneous prime ideals $\mathfrak{p}_1,\ldots,\mathfrak{p}_r \subseteq A$ and integers $k_1,\ldots,k_r$ with $M_\rho/M_{\rho-1} \cong (A/\mathfrak{p}_\rho)(-k_\rho)$, $\rho = 1,\ldots,r$. In particular, $\mathfrak{p}_1\cdots\mathfrak{p}_r M = 0$.*

**Proof** First show that if $M \neq 0$, then it contains a submodule of the isomorphism type $(A/\mathfrak{p})(-k)$, or equivalently, a homogeneous element $0 \neq x \in M$ such that the annihilator ideal $\operatorname{Ann}_A x := \{a \in A \mid ax = 0\} = \mathfrak{p}$ is prime. Let $0 \neq x_0 \in M$. If $\operatorname{Ann}_A x_0$ is not prime, then there exist $a,b \in A$ with $ax_0 \neq 0$, $bx_0 \neq 0$ and $abx_0 = 0$. Then $x_1 := bx_0 \neq 0$, $a \in \operatorname{Ann}_A x_1$, $a \notin \operatorname{Ann}_A x_0$ and so $\operatorname{Ann}_A x_0 \subsetneq \operatorname{Ann}_A x_1$. Since $A$ is Noetherian, in finitely many steps, we get an element $x(= x_s) \in M$, $x \neq 0$ with $\operatorname{Ann}_A x$ prime. (one can also directly choose a homogeneous element $0 \neq x \in M$ such that $\operatorname{Ann}_A x$ is maximal element in $\{\operatorname{Ann}_A y \mid 0 \neq y \in M\}$.) Now, we construct the required chain in $M$. If $M \neq 0$, then there exists a submodule $M_1 \cong (A/\mathfrak{p}_1)(-k_1)$. If $M/M_1 \neq 0$, then there exists $M_2/M_1 \subseteq M/M_1$ with $(M_2/M_1) \cong (A/\mathfrak{p}_2)(-k_2)$ and so on. The chain $0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots$ after finitely many steps will end at $M$, since $M$ is Noetherian. (One can also choose a maximal homogeneous submodule $N \subseteq M$ for which the required chain of submodules of $N$ exists. Then prove that $N$ is necessarily $M$.) •

**(2) Poincaré series** Let $M = \bigoplus_{m\in\mathbb{Z}} M_m$ be a finite graded module over the standard graded $K$-algebra $A = \oplus_{m\in\mathbb{N}} A_m = K[t_0,\ldots,t_n]$ with $A_0 = K$ and $t_0,\ldots,t_n \in A_1$. Then $M_m$, $m \in \mathbb{Z}$, are finite dimensional $K$-vector spaces and $M_m = 0$ for $m << 0$. Therefore, the *Poincaré series*

$$\mathcal{P}_M(Z) := \sum_{m\in\mathbb{Z}} (\operatorname{Dim}_K M_m)\, Z^m$$

is well-defined and is a Laurent-series (with coefficients in $\mathbb{N}$). If $K[T_0,\ldots,T_n] \to A = K[t_0,\ldots,t_n]$ is a representation of $A$ as a residue class algebra of a polynomial algebra, then $\mathcal{P}_M$ is same even if $M$ is considered as $K[T_0,\ldots,T_n]$-module.

**(a) Computation rules for Poincaré series** We note the following elementary computational rules for Poincaré series of finite graded $A$-modules: Let $M, M_1,\ldots,M_r$ be a finite graded modules over the standard graded $K$-algebra $A = \oplus_{m\in\mathbb{N}} A_m = K[t_0,\ldots,t_n]$ with $A_0 = K$ and $t_0,\ldots,t_n \in A_1$.

(1) $\mathcal{P}_{M(-k)} = Z^k\,\mathcal{P}_M$ *for all* $k \in \mathbb{Z}$. (2) *If* $0 \to M_r \to \cdots \to M_0 \to 0$ *is an exact sequence with homogeneous homomorphisms of degrees* 0, *then* $\sum_{\rho=0}^{r} (-1)^\rho\,\mathcal{P}_{M_\rho} = 0$. (3) *If* $f \in A_\delta$ *is a homogeneous non-zero divisor for the A-module M of degree* $\delta > 0$, *then* $\mathcal{P}_{M/fM} = (1-Z^\delta)\,\mathcal{P}_M$. (4) *If* $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M$ *is a chain of homogeneous submodules of the A-module M, then* $\mathcal{P}_M = \sum_{\rho=1}^{r} \mathcal{P}_{M_\rho/M_{\rho-1}}$.

[11] The Noetherian property of modules is named after Emmy Noether (1882-1935) who was the first one to discover the true importance of this property. Emmy Noether is best known for her contributions to abstract algebra, in particular, her study of chain conditions on ideals of rings.

[12] Due to Emanuel Lasker (1868 – 1941) and Max Noether (1844-1920), father of Emmy Noether.

**(b)** The following fundamental lemma was already in the work of Hilbert with a complicated proof.

**Lemma** *Let $M$ be a finite graded module over the standard graded $K$-algebra $A = K[t_0,\ldots,t_n]$, $t_0,\ldots,t_n \in A_1$. Then $\mathcal{P}_M = F/(1-Z)^{n+1}$ with a Laurent-polynomial $F \in \mathbb{Z}[Z^{\pm 1}]$.*

If $M \neq 0$, then after cancelling the highest possible power of $(1-Z)$, we get a *unique* representation

$$\mathcal{P}_M = \frac{Q}{(1-Z)^{d+1}}\,,\ d \geq -1$$

with a Laurent-polynomial $Q \in \mathbb{Z}[Z^{\pm 1}]$, $Q(1) \neq 0$. For $M = 0$, $d = -1$ and $Q = 0$.

The partial fraction decomposition is

$$\mathcal{P}_M = \widetilde{Q} + \sum_{i=0}^{d} \frac{c_i}{(1-Z)^{i+1}} \equiv \sum_{i=0}^{d} \frac{c_i}{(1-Z)^{i+1}}$$

with a uniquely determined Laurent-polynomial $\widetilde{Q} \in \mathbb{Z}[Z^{\pm 1}]$ and unique integers $c_0,\ldots,c_d \in \mathbb{Z}$, where we write $G \equiv H$ for two Laurent-series $G$, $H$ if and only if they differ by a Laurent-polynomial.

Now, using the formula $(1-Z)^{-(n+1)} = \sum_m \binom{m+n}{n} Z^m$ which can be proved directly by differentiating (termwise) $n$-times the geometric series $(1-Z)^{-1} = \sum_m Z^m$, we get:

For $m >> 0$ (more precisely for $m > \deg \widetilde{Q}$), we have

$$\operatorname{Dim}_K M(m) = \chi_M(m) := \sum_{i=0}^{d} c_i \binom{m+i}{i} \quad \text{for } m >> 0\,,$$

where $\chi_M$ is a polynomial function of degree $d$ and in particular, if $d \geq 0$, then

$$\operatorname{Dim}_K M(m) = \chi_M(m) \ \sim\ c_d \cdot \frac{m^d}{d!} = O(m^d) \ \text{ for } m \to \infty\,.$$

where $O$ is the "Big O" symbol [13] and $\sim$ denote the asymptotic equality. The case $d=-1$ is characterized by $\operatorname{Dim}_K M_m = 0$ for $m >> 0$, or by $\operatorname{Dim}_K M = \sum_{m\in\mathbb{Z}} \operatorname{Dim}_K M_m = Q(1) < \infty$.

**(3) Hilbert series** Incidentally, instead of Poincaré-series it is comfortable to consider the *Hilbert-series*

$$\mathcal{H}_M = \sum_{m\in\mathbb{Z}} h_M(m) Z^m = \mathcal{P}_M/(1-Z) \equiv \sum_{i=0}^{d+1} e_i/(1-Z)^{i+1}$$

with the *Hilbert-Samuel function* $h_M : \mathbb{Z} \to \mathbb{N}$:

$$h_M(m) = \sum_{k\leq m} \operatorname{Dim}_K M_m = \operatorname{Dim}_K \Big(\bigoplus_{k\leq m} M_k\Big)$$

and put $e_i := c_{i-1}$, if $i > 0$, and $e_0 := \widetilde{Q}(1)$. For large $m >> 0$, the values $h_M(m)$ are equal to the values of the *Hilbert-Samuel Polynomial*

$$H_M(m) = \sum_{i=0}^{d+1} e_i \binom{m+i}{i} \sim e_{d+1} \cdot m^{d+1}/(d+1)! = O(m^{d+1}) \quad .$$

The integer $d$ is an approximate measure of the size of $M$. For example, if $M = A = P = K[T_0,\ldots,T_n]$, $n \in \mathbb{N}$, then $\mathcal{P}_{K[T_0,\ldots,T_n]} = 1/(1-Z)^n$ and $\mathcal{H}_{K[T_0,\ldots,T_n]} = 1/(1-Z)^{n+1}$.

**(4) Dimension and Multiplicity** The integer $d$ is called the (*projective*) *dimension* $\operatorname{pd}(M)$ and $d+1$ is the (*affine* or *Krull-*) *dimension* $\operatorname{d}(M)$ of the graded module $M$. The integer $\operatorname{e}(M) := e_{d+1} = e_{\operatorname{d}(M)} (= c_{\operatorname{pd}(M)})$ if $\operatorname{pd}(M) \geq 0$) is called the *multiplicity of the graded module $M$* if $\operatorname{pd}(M) \geq 0$. Note that $\operatorname{e}(M) > 0$ if $M \neq 0$. If $M = 0$, then $\operatorname{d}(0) = \operatorname{e}(0) = 0$. If $\mathcal{P}_M = Q/(1-Z)^{\operatorname{pd}(M)}$, then $\mathcal{H}_M = Q/(1-Z)^{1+\operatorname{pd}(M)}$ and $\operatorname{e}(M) = Q(1)$.

In particular, the projective dimension $\operatorname{pd}(K[T_0,\ldots,T_n]) = n$, the affine dimension $\operatorname{d}(K[T_0,\ldots,T_n]) = n+1$ and the multiplicity $\operatorname{e}(K[T_0,\ldots,T_n]) = 1$.

The following computational rules for $\operatorname{d}(M)$ and $\operatorname{e}(M)$ are easy to verify by using the computational rules for Poincaré series given in (2) (a):

[13] The symbol "Big O" is first introduced by the number theorist Paul Bachmann (1837-1920) in 1894. Another number theorist Edmund Landau (1877-1938) adopted it and was inspired to introduce the "small o" notation in 1909. These symbols describe the limiting behaviour of a function. More precisely: For $\mathbb{R}$-valued functions $f$, $g : U \to \mathbb{R}$ defined on some subset $U \subseteq \mathbb{R}$, one writes: (i) $f(x) = O(g(x))$ as $x \to \infty$ ($|f|$ is bounded above by $|g|$, up to constant factor, asymptotically) if there exists a constant $M > 0$ and a real number $x_0 \in \mathbb{R}$ such that $|f(x)| \leq M|g(x)|$ for all $x \geq x_0$, or equivalently $\limsup_{x\to\infty} |f(x)/g(x)| < \infty$. (ii) $f(x) = o(g(x)$ as $x \to \infty$ ($f$ is dominated by $g$ asymptotically) if $\lim_{x\to\infty} |f(x)/g(x)| = 0$.

**(a) Computational rules for dimension and multiplicity** *Let $K$ be a field and $A = \bigoplus_{n\in\mathbb{N}} A_n$ be standard graded $K$-algebra. Then for a finite graded $A$-module $M$, we have :*

(1) $\mathrm{d}(M) = \mathrm{d}(M(-k))$ *and* $\mathrm{e}(M) = \mathrm{e}(M(-k))$, $k \in \mathbb{Z}$.

(2) *Let* $0 \to M_r \to M_{r-1} \to \cdots \to M_0 \to 0$ *be an exact sequence of homogeneous homomorphisms. Then*

$$\sum_{\rho\,,\mathrm{d}(M_\rho)=d} (-1)^\rho \mathrm{e}(M_\rho) = 0, \textit{ where } d := \max{}_{0\le\rho\le r}\{\mathrm{d}(M_\rho)\}.$$

(3) *Let* $f \in A_\delta$ *be a homogeneous element of degree* $\delta > 0$. *Then* $\mathrm{d}(M/fM) \ge \mathrm{d}(M) - 1$. *Moreover, if $f$ is a non-zero divisor for $M$ and $M \neq 0$, then* $\mathrm{d}(M/fM) = \mathrm{d}(M) - 1$ *and* $\mathrm{e}(M/fM) = \delta \cdot \mathrm{e}(M)$.

(4) (*Associativity formula*) *Let* $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$ *be a chain of graded $A$-submodules of $M$.*

$$\textit{Then } d := \mathrm{d}(M) = \max{}_{1\le\rho\le r}\{\mathrm{d}(M_\rho/M_{\rho-1})\}, \textit{ and } \mathrm{e}(M) = \sum_{\rho\,,\mathrm{d}(M_\rho/M_{\rho-1})=d} \mathrm{e}(M_\rho/M_{\rho-1}).$$

(5) *Moreover, if in (4) there are homogeneous prime ideals* $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ *and integers* $k_1,\ldots,k_r \in \mathbb{Z}$ *with* $M_\rho/M_{\rho-1} \cong (A/\mathfrak{p}_\rho)(-k_\rho)$, $\rho = 1,\ldots,r$, *are as in* Lemma in 4.1 (1) (b), *then* $\mathrm{d}(M) = \max{}_{1\le\rho\le r}\{\mathrm{d}(A/\mathfrak{p}_\rho)\}$ *and* $\mathrm{e}(M) = \sum_{\rho\,,\mathrm{d}(A/\mathfrak{p}_\rho)=\mathrm{d}(M)} \mathrm{e}(A/\mathfrak{p}_\rho)$. *In particular, if $M \neq 0$, then there are prime ideals $\mathfrak{p}_\rho$ with* $\mathrm{d}(A/\mathfrak{p}_\rho) = \mathrm{d}(M)$.

**4.2 Projective algebraic sets** Let $K$ be a field and let $\mathrm{P} := K[T_0,\ldots,T_n]$ be the standard polynomial $K$-algebra with the standard gradation $\mathrm{P} := \bigoplus_{m\in\mathbb{N}} \mathrm{P}_m$. Let

$$\mathbb{P}_{\mathrm{P}}(K) := \mathbb{P}^n(K) = \left(K^{n+1}\smallsetminus\{0\}\right)/\sim\, = \{\langle\tau\rangle = \langle\tau_0,\ldots,\tau_n\rangle \mid \tau = (\tau_0,\ldots,\tau_n) \in K^{n+1}\setminus\{0\}\}$$

be the quotient space of the equivalence relation $\sim$ on the set $\left(K^{n+1}\smallsetminus\{0\}\right)$ defined by $\tau = (\tau_0,\ldots,\tau_n) \sim \sigma = (\sigma_0,\ldots,\sigma_n)$ if there exists $\lambda \in K^\times$ such that $\tau_i = \lambda\sigma_i$ for all $i = 0,\ldots,n$. This is called the *$n$-dimensional projective space over $K$*.

For a standard graded $K$-algebra $A = \bigoplus_{m\in\mathbb{N}} A_m = K[t_0,...,t_n]$ with $t_0,...,t_n \in A_1$, let $\mathfrak{A}$ be the kernel of the substitution homomorphism $\varepsilon : K[T_0,...,T_n] \to A$, $T_i \mapsto t_i$, $i = 0,...,n$. Then $\varepsilon$ induces a homogeneous $K$-algebra isomorphism $P/\mathfrak{A} \xrightarrow{\sim} A$ and set of common zeroes

$$\mathbb{P}_K(A) = \mathrm{V}_+(\mathfrak{A}) := \{\langle\tau\rangle \in \mathbb{P}^n(K) \,|F(\tau) = 0 \text{ for all homogeneous } F \in \mathfrak{A}\} \subseteq \mathbb{P}^n(K)$$

of the homogeneous relation ideal $\mathfrak{A}$ in $\mathbb{P}^n(K)$, is called the *projective algebraic set $\mathbb{P}_A(K)$ of $K$-valued points*. Further, if $F_1,\ldots,F_m \in \mathfrak{A}$ is a homogeneous system of generators for $\mathfrak{A}$, then

$$\mathbb{P}_A(K) = \mathrm{V}_+(F_1,\ldots,F_m) = \{\langle\tau\rangle \in \mathbb{P}^n(K) \mid F_i(\tau) = 0\,, i = 1,\ldots,m\}.$$

It is easy to see that the description of $\mathbb{P}_A(K)$ is independent of the representation $A \xrightarrow{\sim} P/\mathfrak{A}$. If $f \in A$ is a homogeneous element with a homogeneous representative $F \in P$, then the zero set

$$\mathrm{V}_+(f) = \{\langle\tau\rangle \in \mathbb{P}_A(K) \mid F(\tau) = 0\}$$

of $f$ in $\mathbb{P}_A(K)$ is well-defined. In particular, for a homogeneous ideal $\mathfrak{a} \subseteq A$, generated by homogeneous elements $f_1,\ldots,f_r \in A$, we have the representation :

$$\text{(4.2.1)} \qquad \mathbb{P}_{A/\mathfrak{a}}(K) = \mathrm{V}_+(f_1,\ldots,f_r) = \bigcap_{\rho=1}^{r} \mathrm{V}_+(f_\rho) \subseteq \mathbb{P}_A(K)\,,$$

Now we prove the following very important useful lemma :

**4.3 Lemma** *Let $K$ be a field and let* $\mathrm{P} := K[T_0,\ldots,T_n]$ *be the standard polynomial $K$-algebra with the standard gradation.*

**(a)** *For a point* $\langle\tau\rangle = \langle\tau_0,\ldots,\tau_n\rangle \in \mathbb{P}^n(K)$, *the vanishing ideal*

$$\mathfrak{P}_{\langle\tau\rangle} := \langle\{F \in \mathrm{P} \mid F \textit{ is a homogeneous polynomial in } \mathrm{P} \textit{ with } F(\tau) = 0\}\rangle$$

*generated by the homogeneous polynomials which vanish on $\langle\tau\rangle$, is a homogeneous prime ideal in* $\mathrm{P}$ *with* $\mathrm{P}/\mathfrak{P}_{\langle\tau\rangle} \xrightarrow{\sim} K[T]$ *a standard graded polynomial algebra in one indeterminate $T$. In particular, the projective dimension* $\mathrm{d}(P/\mathfrak{P}_{\langle\tau\rangle}) = 0$ *and the multiplicity* $\mathrm{e}(P/\mathfrak{P}_{\langle\tau\rangle}) = 1$.

**(b)** *If $\mathfrak{P} \subseteq \mathrm{P}$ is a homogeneous prime ideal with $\mathrm{d}(P/\mathfrak{P}) = 0$ and $\mathrm{e}(P/\mathfrak{P}) = 1$, then there exists a unique point $\langle \tau \rangle \in \mathbb{P}^n(K)$ such that $\mathfrak{P} = \mathfrak{P}_{\langle \tau \rangle}$.*

**Proof** **(a)** We may assume that $\tau_0 = 1$. It is easy to verify that $\mathfrak{P}_{\langle \tau \rangle}$ is generated by $\tau_j T_i - \tau_i T_j$, $0 \leq i, j \leq n$, $i \neq j$ and that the surjective $K$-algebra homomorphism $\mathrm{P} \to K[T]$ defined by $T_0 \mapsto T$ and $T_i \mapsto \tau_i T$, $i = 1, \ldots, n$, has the kernel $\mathfrak{P}_{\langle \tau \rangle}$ and hence $P/\mathfrak{P}_{\langle \tau \rangle} \cong K[T]$.
**(b)** Let $\mathfrak{P} \subseteq \mathrm{P}$ be a homogeneous prime ideal with $\mathrm{d}(P/\mathfrak{P}) = 0$ and $\mathrm{e}(P/\mathfrak{P}) = 1$. Then the $K$-subspace $\mathfrak{P}_1 \subseteq P_1$ is of codimension 1, since $\mathrm{d}(P/\mathfrak{P}) = 0$ and $1 = \mathrm{e}(P/\mathfrak{P}) \geq \mathrm{Dim}_K(\mathrm{P}/\mathfrak{P})_m$ for every $m \in \mathbb{N}$. Therefore $\mathfrak{P} = \langle \mathfrak{P}_1 \rangle = \mathfrak{P}_{\langle \tau \rangle}$ for a unique point $\langle \tau \rangle \in \mathbb{P}^n(K)$. •

For a graded ring $A = \bigoplus_{m \in \mathbb{N}} A_m$, the set of homogeneous prime ideals is denoted by h-Spec $A$.

**4.4 Corollary** *For a standard graded $K$-algebra $A = \bigoplus_{m \in \mathbb{N}} A_m = K[t_0, \ldots, t_n]$ with $t_0, \ldots, t_n \in A_1$, and the substitution homomorphism $\varepsilon : K[T_0, \ldots, T_n] \to A$, $T_i \mapsto t_i$, $i = 0, \ldots, n$. Let $\mathfrak{A} = \mathrm{Ker}\, \varepsilon$. Then the map*

$$\mathbb{P}_A(K) \longrightarrow \{\mathfrak{p} \in \text{h-Spec}\, A \mid \mathrm{d}(A/\mathfrak{p}) = 0 \text{ and } \mathrm{e}(A/\mathfrak{p}) = 1\}, \langle \tau \rangle \longmapsto \mathfrak{p}_{\langle \tau \rangle} := \mathfrak{P}_{\langle \tau \rangle}/\mathfrak{A}$$

*is bijective.*

**Proof** Immediate from the Lemma 4.3, since $\langle \tau \rangle \in \mathbb{P}_A(K)$ if and only if $\mathfrak{A} \subseteq \mathfrak{P}_{\langle \tau \rangle}$. •

**4.5 Lemma** *Let $K$ be a field and $C = \bigoplus_{m \in \mathbb{N}} C_m$ be a standard graded $K$-algebra such that $C$ is an integral domain with $\mathrm{pd}(C) = 0$. Then there exists a finite field extension $L|K$ such that the multiplicity $\mathrm{e}(C)$ is equal to $[L : K]$.*

**Proof** Since $C$ is a standard graded $K$-algebra, $C_1 \neq 0$. Choose $t \in C_1$. Then, since $tC_m \subseteq C_{m+1}$ for all $m \in \mathbb{N}$ and $t$ is a non-zero divisor in $C$, the numerical function $m \mapsto \mathrm{Dim}_K C_m$, is monotone increasing and hence is stationary with the value $\mathrm{e}(C) = \mathrm{Dim}_K C_m$ for large $m >> 0$. But, then the there exists a unique integer $s \in \mathbb{N}$ such that the ascending chain of finite dimensional $K$-vector spaces $C_0 = K \subsetneq C_1/t \subsetneq C_2/t^2 \subsetneq \cdots \subsetneq C_s/t^s = C_{s+1}/t^{s+1} = \cdots$ is stationary and hence $L := C_s/t^s$ is an integral domain which is a finite $K$-algebra of the dimension $\mathrm{Dim}_K C_s = \mathrm{e}(C)$. Therefore $L$ is a finite field extension of $K$ with $[L : K] = \mathrm{e}(C)$. •

First note the following classical Hilbert's Nullstellensatz for algebraically closed field (see [20, § 3]) :

**4.6 Theorem** (Hilbert's Nullstellensatz) *Let $K$ be an algebraically closed field and $A$ be a standard graded $K$-algebra of projective dimension $d = \mathrm{pd}(A) \geq 0$. Further, let $f_1, \ldots, f_r \in A$ be homogeneous elements of positive degrees, $r \leq d$. Then $f_1, \ldots, f_r$ have a common zero in $\mathbb{P}_A(K)$,* i. e., $\emptyset \neq \mathbb{P}_{A/\mathfrak{a}}(K) = \mathrm{V}_+(f_1, \ldots, f_r) \subseteq \mathbb{P}_A(K)$, *where $\mathfrak{a} := Af_1 + \cdots + Af_r$.*

**Proof** By induction on $d$ and $r$. If $d = 0$, then $r = 0$. By Lemma in 4.1 (1) (b) there exists a homogeneous prime ideal $\mathfrak{p} \subseteq A$ with $\mathrm{d}(A/\mathfrak{p}) = 0$. By Lemma 4.5, necessarily $\mathrm{e}(A/\mathfrak{p}) = 1$, since $K$ is algebraically closed and hence $\mathfrak{p}$ defines — by Corollary 4.4 — a point in $\mathbb{P}_A(K)$.
For the inductive step from $d$ to $d+1$, consider a prime ideal $\mathfrak{p} \subseteq A$ with $d = \mathrm{d}(A/\mathfrak{p})$. It is enough to prove that $\emptyset \neq V_+(\overline{f}_1, \ldots, \overline{f}_r) \subseteq \mathbb{P}_{A/\mathfrak{p}}(K) \subseteq \mathbb{P}_A(K)$, where $\overline{f}_1, \ldots, \overline{f}_r$ denote the residue classes of $f_1, \ldots, f_r$ in $A/\mathfrak{p}$. We may therefore assume that $A$ is an integral domain and $f_r \neq 0$. Then $\mathrm{d}(A/Af_r) = d - 1$. By induction hypothesis it follows $\emptyset \neq V_+(\overline{f}_1, \ldots, \overline{f}_{r-1}) = V_+(f_1, \ldots, f_r) \subseteq \mathbb{P}_{A/Af_r}(K)$, where now $\overline{f}_1, \ldots, \overline{f}_{r-1}$ are the residue classes in $A/Af_r$. •

Now, we prove the analog of the above Theorem 4.6 for 2-fields. For this, first we recall definition and some basic results for 2-fields. The only property of the field $\mathbb{R}$ of real numbers which will be used in the following is : *every polynomial of odd degree with coefficients in $\mathbb{R}$ has a zero in $\mathbb{R}$.* We would like to formulate this property axiomatically :

**4.7 Definition** A field $K$ is called a 2-*field* if every polynomial $F \in K[X]$ of odd degree has a zero in $K$. The 2-fields are defined in [35]. For example, the fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers are 2-fields. More generally, algebraically closed fields are 2-fields and every real closed field is a 2-field, see 3.1 (3) (e).

The following elementary characterization of 2-fields is useful :

**4.8 Lemma** *For a field $K$, the following statements are equivalent*: (i) *$K$ is a 2-field.* (ii) *If $\pi \in K[X]$ is a prime polynomial of degree $> 1$, then the* $\deg \pi$ *is even.* (iii) *If $L|K$ is a non-trivial finite field extension of $K$, then the degree $= [L:K] = \mathrm{Dim}_K L$ is even.*

**Proof** The reader is recommended to prove the implications : (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii)$\Rightarrow$(i) . •

Now, we shall prove the analogue of the Theorem 4.6 — Hilbert's Nullstellensatz for 2-fields.

**4.9 Theorem** ( Hilbert's Nullstellensatz for 2-fields — U. Storch, 2003 ) *Let $K$ be a 2-field and $A$ be a standard graded $K$-algebra of projective dimension $d = \mathrm{pd}(A) \geq 0$ and of odd multiplicity $\mathrm{e}(A)$. Further, let $f_1, \ldots, f_r \in A$ be homogeneous elements of positive odd degrees, $r \leq d$. Then $f_1, \ldots, f_r$ have a common zero in $\mathbb{P}_A(K)$,* i. e. $\emptyset \neq \mathbb{P}_{A/\mathfrak{a}}(K) = \mathrm{V}_+(f_1, \ldots, f_r) \subseteq \mathbb{P}_A(K)$, $\mathfrak{a} := Af_1 + \cdots + Af_r$.

**Proof** For $M := A$, let $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M$ be a chain with $M_\rho / M_{\rho-1} = (A/\mathfrak{p}_\rho)(-k_\rho)$ as in Lemma in 4.1 (1) (b). Then by 4.1 (4) (a) (5) we have $\mathrm{e}\,(A) = \sum_{\rho\,,\,\mathrm{d}\,(A/\mathfrak{p}_\rho) = d} \mathrm{e}\,(A/\mathfrak{p}_\rho)$. Since the multiplicity $\mathrm{e}(A)$ is odd by assumption, it follows that at least one of $\mathrm{e}\,(A/\mathfrak{p}_\rho)$ with $\mathrm{pd}\,(A/\mathfrak{p}_\rho) = d$ is also odd. If $d = 0$, then by Lemma 4.5 and Lemma 4.8 necessarily $\mathrm{e}\,(A/\mathfrak{p}_\rho) = 1$ for one $\mathfrak{p}_\rho$ with $\mathrm{d}\,(A/\mathfrak{p}_\rho) = 0$, and such a prime ideal $\mathfrak{p}_\rho$ defines a point in $\mathbb{P}_A(K)$. For the inductive step from $d$ to $d+1$, we may assume that $A$ is an integral domain and $f_r \neq 0$. Then $\mathrm{e}\,(A/Af_r) = \mathrm{e}\,(A) \cdot \deg f_r$ is also odd and $\mathrm{d}\,(A/Af_r) = d$, and by applying the induction hypothesis to $A/Af_r$ and the residue classes $\overline{f}_1, \ldots, \overline{f}_{r-1}$, the assertion follows. •

For $A = K[T_0, \ldots, X_n]$, we have $\mathrm{d}(A) = n$, $\mathrm{e}(A) = 1$ and $\mathbb{P}_A(K) = \mathbb{P}^n(K)$. This special case of the Theorem 4.9 was already proved by Albrecht Pfister in [35] as Theorem 3 :

**4.10 Corollary** ( Projective Nullstellensatz for 2-fields ) *Let $K$ be a 2-field. Then homogeneous polynomials $f_1, \ldots, f_r \in K[T_0, \ldots, T_n]$, $r \leq n$ of odd degrees have a common non-trivial zero in $K^{n+1}$.*

Since the field $\mathbb{R}$ is a 2-field, in particular, we have :

**4.11 Corollary** ( Real Projective Nullstellensatz ) *Homogeneous polynomial polynomials $f_1, \ldots, f_n \in \mathbb{R}[T_0, \ldots, T_n]$ of odd degrees over $\mathbb{R}$ have a common non-trivial zero in $\mathbb{R}^{n+1}$.*

We use the Real Projective Nullstellensatz 4.11 to provide an algebraic proof of the well-known Borsuk-Ulam theorem which states that :

**4.12 Theorem** ( Borsuk-Ulam [14] ) *For every continuous map $g : \mathrm{S}^n \to \mathbb{R}^n$, $n \in \mathbb{N}$, there exist anti-podal points $t$, $-t \in \mathrm{S}^n$ with $g(t) = g(-t)$.*

The proof of Borsuk-Ulam theorem for the case $n = 1$ is an easy application of the intermediate value theorem. The case $n = 2$ is already non-trivial and its needs the concept of the first fundamental group which was introduced by Henri Poincaré (1854–1912) — who was responsible for formulating the Poincaré conjecture. The general case is usually proved by using higher homology groups.

[14] This was conjectured by Stanislaw Ulam (1909–1984) and was proved by Karol Borsuk (1905 – 1982) in 1933 by elementary methods but technically involved. Borsuk presented the theorem at the International congress of mathematics at Zürich in 1932 and was published in the *Fundamentae Mathematicae* **20**, 177-190 (1933) with the title *Drei Sätze über n-dimentionale euclidische Sphäre.*

Borsuk-Ulam Theorem is fascinating even today. It implies the classical Theorem of Brouwer [15] and the Invariance of Dimension Theorem [16].

**Proof** Recall that $S^n = \{t = (t_0,\ldots,t_n) \in \mathbb{R}^{n+1} \mid \|t\|^2 = \sum_{i=0}^n t_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$ is the $n$-sphere. Consider the odd continuous map $f : S^n \to \mathbb{R}^n$, $t \mapsto f(t) := g(t) - g(-t)$ and the Borsuk-Ulam's Nullstellensatz see (i) in Theorem 4.13 below. •

We now prove the Borsuk-Ulam's Nullstellensatz and its equivalent statements :

**4.13 Theorem** *Let $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) (Borsuk-Ulam's Nullstellensatz) *Every continuous odd map* [17] $f : S^n \longrightarrow \mathbb{R}^n$, *$n \in \mathbb{N}$, has a zero.*

(ii) (Borsuk's antipodal theorem) *Every continuous map $h : \overline{B}^n \longrightarrow \mathbb{R}^n$ with $n \geq 1$ and the restriction $h|_{S^{n-1}} : S^{n-1} \longrightarrow \mathbb{R}^n$ odd, has a zero.*

(iii) (Real Projective Nullstellensatz — Corollary 4.11 ) *Homogeneous polynomials $f_1, \ldots, f_n \in \mathbb{R}[T_0,...,T_n]$ of odd degree have a common non-trivial zero in $\mathbb{R}^{n+1}$.*

**Proof** (i) $\Longleftrightarrow$ (ii) Note that for $n \geq 1$, the odd continuous maps $f : S^n \to \mathbb{R}^n$ correspond to the continuous maps $h : \overline{B}^n \to \mathbb{R}^n$ such that the restriction $h|_{S^{n-1}} : S^{n-1} \to \mathbb{R}^n$ of $h$ to the subset $S^{n-1} \subseteq \overline{B}^n$ is odd. For a given $f : S^n \to \mathbb{R}^n$ define $h(t) := f(\sqrt{1-\|t\|^2}, t)$, $t \in \overline{B}^n$, and conversely for a given $h : \overline{B}^n \to \mathbb{R}^n$ define $f(t_0,t) := h(t)$, if $t_0 \geq 0$, $f(t_0,t) := -h(-t)$, if $t_0 \leq 0$, $(t_0,t) \in S^n \subseteq \mathbb{R} \times \mathbb{R}^n$.

(i) $\Rightarrow$ (iii) From (i) in particular, it follows that $n$ odd polynomial functions $f_1, \ldots, f_n : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ have a common zero on $S^n$. If $F \in \mathbb{R}[T_0, \ldots, T_n]$ defines an odd polynomial function $F : \mathbb{R}^{n+1} \to \mathbb{R}$, then all homogeneous components of even degree in $F$ are zero, i .e. in the homogeneous decomposition of $F$ only odd degree homogeneous components can occur. Suppose that $F = \sum_{i=0}^m F_{2i+1}$, $F_{2m+1} \neq 0$, is the homogeneous decomposition of $F$ with homogeneous components $F_1, \ldots, F_{2m+1}$ of odd degrees $1, \ldots, 2m+1$, respectively. Now, observe that $F$ and the *homogeneous* polynomial $Q^m F_1 + Q^{m-1} F_3 + \cdots + F_{2m+1}$, $Q := T_0^2 + \cdots + T_n^2$, have the same values on the sphere $S^n$.

(iii) $\Rightarrow$ (i) : Let $f = (f_1, \ldots, f_n) : S^n \to \mathbb{R}^n$, $n \in \mathbb{N}$, with $f_i : S^n \to \mathbb{R}$, $i = 1, \ldots, n$, odd and continuous. Then by the well-known Weierstrass Approximation Theorem [18] for every $k \in \mathbb{N}^*$, there exist *polynomial functions* $g_{ik}$ with $|g_{ik}(t) - f_i(t)| \leq 1/k$ for $i = 1, \ldots, n$ and all $t \in S^n$. For the odd parts $f_{ik}(t) := (g_{ik}(t) - g_{ik}(-t))/2$, it follows $|f_{ik}(t) - f_i(t)| = \frac{1}{2}|(g_{ik}(t) - f_i(t)) - (g_{ik}(-t) - f_i(-t))| \leq 1/k$. By the Real Algebraic Nullstellensatz 4.11, the $f_{ik}$, $i = 1, \ldots, n$, have a common zero $t_k \in S^n$. Then an accumulation point $t \in S^n$ of $t_k$, $k \in \mathbb{N}^*$, is a common zero of the $f_1, \ldots, f_n$. •

**4.14 Remark** Note that we have proved Real Projective Nullstellensatz in Corollary 4.11 and hence the equivalence in the Theorem 4.13 proves the Borsuk-Ulam's Nullstellensatz 4.13 (ii) also. In particular, we have proved the Borsuk-Ulam Theorem.

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[15] **Theorem** (Brouwer's fixed point theorem – Brouwer L. E. J. (1881-1966)) *Every continuous map $f : \overline{B}^n \longrightarrow \overline{B}^n$ of the unit ball $\overline{B}^n := \{t \in \mathbb{R}^n \mid \|t\| \leq 1\}$ has a fixed point.* **Proof** If $f$ has no fixed point, then the continuous map $h : \overline{B}^n \to S^{n-1} \subseteq \mathbb{R}^n$, which maps the point $t \in \overline{B}^n$ to the point of intersection of the line-segment $L(f(t), t) := \{f(t) + \lambda t \mid \lambda \in [0,1]\} \subseteq \mathbb{R}^n$ with the sphere $S^{n-1} \subseteq \overline{B}^n$ has no zero. But, $h|_{S^{n-1}} = \mathrm{id}$ and in particular, $h(-t) = -h(t)$ for $t \in S^{n-1}$. Therefore, for $n = 1$, the Nullstellensatz is equivalent with the *Intermediate Value theorem* : Every continuous map $h : [-1,1] \to \mathbb{R}$ with $h(-1) = -h(1)$ has a zero.

[16] **Theorem** (Invariance of dimension) *For $m > n$, there is no injective continuous map from an open subset $U \subseteq \mathbb{R}^m$ into $\mathbb{R}^n$. In particular, if $m \neq n$, then the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*

[17] A map $f : S^n \longrightarrow \mathbb{R}^n$ is called an *odd map* if $f(x) = -f(-x)$ for every $x \in S^n$.

[18] **Theorem** (Weierstrass) *Let $X \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$ be a compact subset. Then the set of polynomial functions $\mathbb{R}[T_1,...,T_n]$ is dense in $(C(X,\mathbb{R}), \|\cdot\|_{\sup})$, where $C(X,\mathbb{R})$ is the $\mathbb{R}$-algebra of $\mathbb{R}$-valued continuous functions on $X$ and for every $f \in C(X,\mathbb{R})$, $\|f\|_{\sup} := \mathrm{Sup}\{\|f(x)\| \mid x \in X\}$.* ( Karl Weierstrass (1815-1897) is known as the father of modern analysis, and contributed to the theory of periodic functions, functions of real variables, elliptic functions, Abelian functions, converging infinite products, and the calculus of variations. He also advanced the theory of bilinear and quadratic forms. )