RINGS OF MINIMAL MULTIPLICITY

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Abstract. In this exposition, we discuss two theorems of S. S. Abhyankar about Cohen-Macaulay local rings of minimal multiplicity and graded rings of minimal multiplicity.

1. Introduction

We shall discuss two theorems of S. S. Abhyankar about rings of minimal multiplicity which he proved in his 1967 paper *"Local rings of high embedding dimension"* [1] which appeared in the American Journal of Mathematics. The first result gives a lower bound on the multiplicity of the maximal ideal in a Cohen-Macaulay local ring and the second result gives a lower bound on the multiplicity of a standard graded domain over an algebraically closed field.

The origins of these results lie in projective geometry. Let k be an algebraically closed field and X be a projective variety. We say that it is *non-degenerate* if it is not contained in a hyperplane. Let $I(X)$ be the ideal of X. It is a homogeneous ideal of the polynomial ring $S = k[x_0, x_1, \ldots, x_r]$. The homogeneous coordinate ring of X is defined as $R = S(X) = S/I(X)$. Then R is a graded ring and we write it as $R = \oplus_{n=0}^{\infty} R_n$. Here $R_n = S_n/(I(X) \cap S_n)$. The Hilbert function of R is the function $H_R(n) = \dim_k R_n$.

Theorem 1.1 (Hilbert-Serre). *There exists a polynomial $P_R(x) \in \mathbb{Q}[x]$ so that for all large n, $H_R(n) = P_R(n)$. The degree of $P_R(x)$ is the dimension d of X and we can write*

$$P_R(x) = e(R)\binom{x+d}{d} - e_1(R)\binom{x+d-1}{d-1} + \cdots + (-1)^d e_d(R).$$

Definition 1.2. *The integer $e(R)$, also denoted by $\deg X$, is called the degree of the projective variety X.*

Theorem 1.3. *If $X \subset \mathbb{P}^r$ is a non-degenerate irreducible projective variety. Then*

$$1 + \operatorname{codim} X \leq \deg X \text{ equivalently } \dim R_1 - \dim R + 1 \leq e(R).$$

Definition 1.4. *We say that X is a projective variety of minimal degree if*

$$\deg X = 1 + \operatorname{codim} X.$$

The graded ring R is said to have minimal multiplicity if

$$\dim R_1 - \dim R + 1 = e(R).$$

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Del Pezzo classified all projective surfaces of minimal degree in 1886. Bertini gave the classification for all projective varieties of minimal degree. Simpler and modern treatments for this classification have been given by Nagata [6], Harris [4], Xambó [8] and Eisenbud-Harris [3].

2. The Dimension and depth of Noetherian local rings

1. **Dimension of a Noetherian ring.** Let R be a Noetherian ring. A chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_g$$

where all the prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_g$ are distinct is said to have length g. The *Krull dimension* of R is defined to be

$$\dim R = \sup\{g \mid R \text{ has a chain of prime ideals of length } g\}.$$

If $\mathfrak{P}$ is a prime ideal of R then its height is defined to be

$$\operatorname{ht} \mathfrak{P} = \sup\{g \mid \text{ there is a chain of length } g \text{ of prime ideals contained in } \mathfrak{P}\}.$$

Theorem 2.1. *Suppose K is a field and $S = K[X_1, X_2, \ldots, X_n]$ is the polynomial ring. Let $\mathfrak{P}$ is a prime ideal of S and $R = S/\mathfrak{P}$. Let L be the quotient field of R. Then*

$$\dim R = \operatorname{trdeg}_K L.$$

Now let $(R, \mathfrak{m})$ be a local ring. Then $\dim R = \operatorname{ht} \mathfrak{m}$. The Krull's Altitude Theorem asserts that $\dim R \leq \mu(\mathfrak{m})$ where $\mu(\mathfrak{m})$ is the minimum number of generators required for $\mathfrak{m}$. There are two more ways to define the Krull dimension of R. The *Chevalley dimension* of R is defined as

$$c(R) = \inf\{n \mid \text{ there exist } n \text{ elements } a_1, a_2, \ldots, a_n \in \mathfrak{m} \text{ with } \mathfrak{m} = \sqrt{(a_1, a_2, \ldots, a_n)}\}.$$

The most unintuitive but a very effective way to find the Krull dimension is to use the Hilbert-Samuel function $H_I : \mathbb{N} \to \mathbb{N}$, defined as

$$H_I(n) = \ell(R/I^n).$$

P. Samuel showed that there is a polynomial $P_I(x) \in \mathbb{Q}[x]$, called the *Hilbert polynomial* of I so that for all large n, $P_I(n) = H_I(n)$. Moreover $\deg P_I(x) = d = \dim R$. We write

$$P_I(x) = e_0(I)\binom{x+d-1}{d} - e_1(I)\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

Here $e_0(I), e_1(I), \ldots, e_d(I)$ are integers and they are called the Hilbert coefficients of I. The number $e_0(\mathfrak{m}) := e(R) > 0$ and it is called the *multiplicity of the local ring* $(R, \mathfrak{m})$.

Theorem 2.2 (The Dimension Theorem). *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$\dim R = c(R) = \deg P_I(x).$$

Definition 2.3. *Let M be a finite R-module over a Noetherian ring. Then the dimension of M is defined to be* $\dim M = \dim R/\operatorname{ann}(M)$. *The embedding dimension of a Noetherian local ring is* $\operatorname{emb}\dim R = \mu(\mathfrak{m})$. *If $\dim R = d$ and $(a_1, \ldots, a_d)$ is $\mathfrak{m}$-primary then we say that $a_1, a_2, \ldots, a_d$ is a* system of parameters *for R.*

2. **Depth of a Noetherian local ring.** In order to introduce depth of a local Noetherian ring $(R, \mathfrak{m})$ we introduce the notion of an M-regular sequence where M is a finite R-module. Let I be a proper ideal of R. A sequence $a_1, a_2, \ldots, a_n \in I$ is called an *M-regular sequence* of length n if a_i is a nonzerodivisor on $M/(a_1, a_2, \ldots, a_{i-1})M$ for all $i = 1$.

Theorem 2.4. *The length of any maximal M-regular sequences in I depends only on I and M. It is denoted by* $\operatorname{depth}_I(M)$. *We have*

$$\operatorname{depth}_I(M) \leq \dim M.$$

Definition 2.5. *A local ring $(R, \mathfrak{m})$ is called Cohen-Macaulay if $\mathfrak{m}$ has an R-regular sequence of length* $\dim R$. *A Noetherian ring R is called Cohen-Macaulay if $R_{\mathfrak{m}}$ is Cohen-Macaulay for every maximal ideal $\mathfrak{m}$ of R.*

Cohen-Macaulay rings are ubiquitous. We give several examples.

Example 2.6. Let k be a field and $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring. Macaulay proved that R is Cohen-Macaulay. I. S. Cohen proved that any regular local ring $(R, \mathfrak{m})$ is Cohen-Macaulay. Recall that R is called regular if $\dim R = \operatorname{emb}\dim R$. In particular the power series ring $k[[x_1, \ldots, x_n]]$ and the ring of convergent power series $\mathbb{C}\{x_1, x_2, \ldots, x_n\}$ are regular and hence Cohen-Macaulay.

Theorem 2.7 (Hochster, 1972). *Let k be a field. Let $M_1, M_2, \ldots M_r$ be monomials in $S = k[x_1, x_2, \ldots, x_n]$. Let $R = k[M_1, M_2, \ldots, M_r]$. If R is integrally closed in its quotient field then it is Cohen-Macaulay.*

Theorem 2.8 (Hochster-Roberts, 1974). *Let $R = k[x_1, x_2, \ldots, x_n]$ be a polynomial ring over an algebraically closed field k. Let G be a linearly reductive linear algebraic group over k acting linearly on R. Then the ring of invariants*

$$R^G = \{f \in R \mid g(f) = f \text{ for all } g \in G\}.$$

is a Cohen-Macaulay ring.

Theorem 2.9. *If R is CM then so are $R[x]$ and $R[[x]]$ and R_p for any prime p. If a is a nonzero-divisor in R and R is Cohen-Macaulay then so is $R/(x)$.*

Cohen-Macaulay rings can be characterised in many ways. We shall use a criterion in terms of the multiplicity of a system of parameters. If $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d and $a_1, a_2, \ldots, a_d$ be a system of parameters. Then it is a regular sequence. Let $I = (a_1, a_2, \ldots, a_d)$. One can show that the Hilbert function of I is given by

$$H_I(n) = \ell(R/I^n) = \ell(R/I)\binom{n+d-1}{d}.$$

Hence $e_0(I) = \ell(R/I)$. In fact the converse is also true.

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then the following are equivalent*
(1) *R is Cohen-Macaulay.*
(2) *If $a_1, a_2, \ldots, a_d$ is a system of parameters then it is an R-regular sequence.*
(3) *For some system $a_1, a_2, \ldots, a_d$ of parameters, $e_0(I) = \ell(R/I)$ where $I = (a_1, a_2, \ldots, a_d)$.*

3. Cohen-Macaulay local rings of minimal multiplicity

We shall now prove the first Theorem of Abhyankar in his 1967 paper which gives a lower bound for the multiplicity of a Cohen-Macaulay local ring. We shall use basic facts about reductions of ideals which were introduced by Northcott-Rees in 1954.
Let I be an ideal of a commutative wing R. An ideal $J \subset I$ is called a reduction of I, if there is an $n \in \mathbb{N}$ such that $JI^n = I^{n+1}$. Intuitively J is a simpler version of I. A related notion is that of integral closure of an ideal. We say that $x \in R$ is *integral over* I if x satisfies an equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$$

for some $a_i \in I^i$ for $i = 1, 2, \ldots, n$. The set of elements of R which are integral over I is called the *integral closure* of I. We say that I is integral over J if every element of I is integral over J. If $R/\mathfrak{m}$ is an infinite field then there exists a reduction J of any $\mathfrak{m}$-primary ideal so that J is generated by d elements.
A local ring R is called *quasi-unmixed* if every minimal prime $\mathfrak{p}$ in its $\mathfrak{m}$-adic completion $\hat{R}$ satisfies $\hat{R}/\mathfrak{p} = \dim R$.

Theorem 3.1 (Rees, 1961). *Let $J \subset I$ be $\mathfrak{m}$-primary ideals of a Noetherian local ring R where J is a reduction of I. Then $e(I) = e(J)$. If R is quasi-unmixed and $e(I) = e(J)$ then J is a reduction of I.*

Proof. The converse is a deep theorem of Rees which we will not prove. If J is a reduction of I then $JI^r = I^{r+1}$. Hence $J^n I^r = I^{n+r}$ for all $n \geq 1$. Hence $J^{n+r} \subset I^{n+r} \subset J^n$. Hence for all $r \geq 1$,

$$\ell(R/J^n) \leq \ell(R/I^{n+r}) \leq \ell(R/J^{n+r}).$$

For n large these lengths are given by their respective Hilbert polynomials. Divide by $n^d/d!$ and take limit as $n \to \infty$ to see that $e(I) = e(J)$. □

Theorem 3.2 (Abhyankar, 1967). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Then*

$$\operatorname{emb}\dim(R) - \dim R + 1 \leq e(R).$$

Let $R/\mathfrak{m}$ be infinite. Then $\operatorname{emb}\dim(R) - \dim R + 1 = e(R)$ *if and only if for some and hence all minimal reductions J of* $\mathfrak{m}, J\mathfrak{m} = \mathfrak{m}^2$.

Proof. By passing to the Nagata extension $R(x) = R[x]_{\mathfrak{m}[x]}$ we may assume that the residue field $R/\mathfrak{m}$ is infinite. Let J be a minimal reduction of $\mathfrak{m}$. Then J is generated by d elements of $\mathfrak{m}$. Hence these generators form an R-regular sequence as R is Cohen-Macaulay. Hence

$$\ell(R/\mathfrak{m}) + \ell(\mathfrak{m}/\mathfrak{m}^2) + \ell(\mathfrak{m}^2/J\mathfrak{m}) = \ell(R/J) + \ell(J/\mathfrak{m}J) = e(R) + d.$$

Therefore we have $1 + \operatorname{emb}\dim(R) + \ell(\mathfrak{m}^2/J\mathfrak{m}) = e(R) + d$. Hence $\operatorname{emb}\dim(R) - \dim R + 1 \leq e(R)$. Morever the equality holds if and only of $J\mathfrak{m} = \mathfrak{m}^2$. □

Example 3.3. Let k be a field in the examples below.

1. If R is regular then it has minimal multiplicity. Indeed, $\operatorname{emb}\dim R = \dim R$ and $e(R) = 1$. Hence $\operatorname{emb}\dim R - \dim R + 1 = 1 = e(R)$.
2. Let e be a positive integer and $R = k[[x^e, \ldots, x^{2e-1}]]$. Then $\dim R = 1, \operatorname{emb}\dim R = e$ and $e(R) = e$. Hence it has minimal multiplicity.

(3) Let $R = k[[(X,Y)^s]]$. Then R is Cohen-Macaulay of dimension 2 and $\operatorname{emb\,dim} R = s + 1, e(R) = s$. Hence it has minimal multiplicity. In fact it has a rational singularity.
(4) There exist Cohen-Macaulay local rings of minimal multiplicity which do not have rational singularity. For example $R = k[X,Y,Z]$ and $\mathfrak{m} = (X,Y,Z)$. Then $R_{\mathfrak{m}}/(Z^2 - X^4 - Y^4)$ has minimal multiplicity but it does not have rational singularities.
(5) Let $R = k[[X,Y,Z]]/(X^2+Y^2+Z^2)$. Then R is Cohen-Macaulay with minimal multiplicity since $\operatorname{emb\,dim} R - \dim R + 1 = 3 - 2 + 1 = 2 = e(R)$.

(6) Let $R = k[[X,Y,Z]]/(X^n + Y^n + Z^n)$ where $n \geq 3$. Then R is Cohen-Macaulay without minimal multiplicity. In fact $\operatorname{emb\,dim} R - \dim R + 1 = 3 - 2 + 1 = 2 < n$.

4. Homogeneous domains of minimal multiplicity

In this section we present Abhyankar's second theorem from his 1967 paper. This theorem is a graded analogue of the local theorem. We follow the treatment given by K. Yamagishi and U. Orbanz in the book *"Equimultiplicity and Blowingup"*. We begin by defining homogeneous domains.

Definition 4.1. *Let k be an algebraically closed field and $A = \bigoplus_{n=0}^{\infty} A_n$ be a Noetherian graded domain where $A_0 = k$ and $A = k[A_1]$. Then A is called a homogeneous domain.*

Example 4.2. Let P be a prime ideal of $S = \mathbb{C}[X_0, X_2, \ldots, X_n]$ generated by homogeneous polynomials. Then S/P is a homogeneous domain.

Let $\mathfrak{m} = A_+ = \bigoplus_{n=1}^{\infty} A_n$. Then A_+ is the unique maximal homogeneous ideal of A. The Hilbert function of A is defined to be $H_A(n) = \dim_k A/\mathfrak{m}^n$. By Hilbert-Serre Theorem, $H_A(n)$ is given by a polynomial $P_A(x) \in \mathbb{Q}[x]$ for all large n. This is called the Hilbert polynomial of A. Moreover $d = \dim A = \deg P_A(x)$. We write the Hilbert polynomial as

$$P_A(x) = e(A)\binom{x+d-1}{d} - e_1(A)\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(A).$$

The coefficient $e(A)$ is called the degree or the multiplicity of A. The embedding dimension of A is defined to be $\dim A_1$.

Theorem 4.3. (1) *Let A be a homogeneous domain. Then* $\operatorname{emb\,dim} A - \dim A + 1 \leq e(A)$. (2) *Let* $\dim A \geq 2$. *If* $\operatorname{emb\,dim} A - \dim A + 1 = e(A)$ *then A is normal and Cohen-Macaulay.*

Proof. We give the proof assuming that $\dim A = 2$. Let $\overline{A}$ be the integral closure of A in its quotient field. Then $\overline{A}$ is graded and it has a unique maximal homogeneous ideal $\mathfrak{m}$. The canonical map

$$A_1 \to \mathfrak{m}/\mathfrak{m}^2$$

is an injective linear transformation of k-vector spaces. Then $B = \overline{A}$ is a two-dimensional normal domain. Hence it is Cohen-Macaulay. By the projection formula for the multiplicities we have

$$e(A) = e(A_1 B) \geq e(B_+) \geq \operatorname{emb\,dim} B - \dim B + 1.$$

Therefore we have

$$\operatorname{emb\,dim} A - \dim A + 1 \leq \operatorname{emb\,dim} B - \dim B + 1 \leq e(B_+) \leq e(A_+).$$

If equality holds in this equation then $\mathfrak{m}$ is generated by A_1. Hence $\overline{A} = k[A_1] = A$. Hence A is normal and therefore it is Cohen-Macaulay. The general case is treated by an application of Bertini's theorem. ∎

Definition 4.4. *If* A *satisfies* $\operatorname{emb}\dim A - \dim A + 1 = e(A)$ *then* A *is said to have minimal multiplcity.*

Example 4.5. (1) The graded domain $A = \mathbb{C}[X^n, X^{n-1}Y, \ldots, Y^n]$ is a homogeneous domain of minimal multiplicity. Note that

$$\operatorname{emb}\dim A - \dim A + 1 = n + 1 - 2 + 1 = n = e(A).$$

(2) The ring $A = \mathbb{C}[x^2, xy, y]$ is Cohen-Macaulay, non-normal and it satisfies

$$\operatorname{emb}\dim A - \dim A + 1 = e(A) = 2.$$

(3) Consider the graded ring $B = \mathbb{C}[x, y, z]/(x^2) \cap (y, z)$. This is not a domain. It satisfies Abhyankar's equality since

$$\operatorname{emb}\dim A - \dim A + 1 = 3 - 2 + 1 = 2.$$

But A is not Cohen-Macaulay.

(4) Let $C = \mathbb{C}[x, y, z, w]/(x, y) \cap (z, w)$. Then C is reduced but not a domain. It does not satisfy Abhyankar's inequality since

$$\operatorname{emb}\dim C - \dim C + 1 = 3 > e(C) = 2.$$

References

[1] S. S. Abhyankar, *Local rings of high embedding dimension,* American Journal of Mathematics **89** (1967),1073-1077. 1

[2] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity,* J. Algebra **88** (1984), 89-133.

[3] D. Eisenbud and J. Harris, *On varieties of minimal degree, A centennial account,* Proceedings of Symposia in Pure Mathematics **46** (1987), 3-13.

[4] J. Harris, *A bound on the geometric genus of projective varieties,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 8 (1981), 35–68.

[5] M. Herrmann and U. Orbanz, Equimultiplicty and Blowing up, An algebraic study. With an appendix by B. Moonen. Springer-Verlag, Berlin, 1988.

[6] M. Nagata, *On rational surfaces I: Irreducible curves of arithmetic genus 0 or 1,* Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **32** (1960), 351–370.

[7] J. D. Sally, *Cohen-Macaulay local rings of maximal embedding dimension,* J. Algebra **56** (1979), 168–183.

[8] S. Xambo, *On projective varieties of minimal degree,* Collect. Math. **32** (1981), 149-163.

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