

Borsuk-Ulam Theorem and Hilbert's Nullstellensatz — and vice versa

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Abstract

A well-known Borsuk-Ulam theorem says that:

Theorem (Borsuk-Ulam) *Every continuous map $f : S^n \rightarrow \mathbb{R}^n$, $n \in \mathbb{N}$, takes identical values on a pair of antipodal points.*

The main aim of talk is to provide an algebraic proof this theorem by using projective algebraic sets. In fact, we prove a more general form of Borsuk-Ulam theorem called the Borsuk's Nullstellensatz by establishing its equivalence with the real algebraic Nullstellensatz.

The proof of Borsuk-Ulam theorem for the case $n = 1$ is an easy application of the intermediate value theorem. The case $n = 2$ is already non-trivial and its needs the concept of the first fundamental group which was introduced by Poincaré. The general case is usually proved by using higher homology groups.

Our proof of the real algebraic Nullstellensatz is elementary and uses standard definitions and basic properties of Poincaré series, projective dimension and multiplicity of (standard) graded algebras over a field. From these considerations our proof of the real algebraic Nullstellensatz only uses the fundamental property of the real numbers, namely: every odd degree polynomial over the field of real numbers has a real root. We say a field K is a 2-field if every odd degree polynomial over K has a root in K . Therefore the real algebraic Nullstellensatz can be generalized to an arbitrary 2-field K . More precisely, we prove :

Theorem *Let A be a standard graded algebra over a 2-field K of projective dimension $d \geq 0$ with odd multiplicity e and $f_1, \dots, f_r \in A$ be homogeneous elements of odd degree with $r \leq d$. Then f_1, \dots, f_r have a common zero in the projective space $\mathbb{P}_A(K)$ of A over K .*

¹On the occasion of Birthday of Late Professor Shreeram S. Abhyankar, Founder Director, Bhaskaracharya Pratishthana, Pune.