Lebesgue Measure

R. R. Simha

1 Preliminaries

Definition 1.1 *A σ-algebra Σ on a set X is a set Σ of subsets of X which is closed under countable unions and complementation, i.e.* $E \in \Sigma \Longrightarrow X \setminus E \in \Sigma$, *and* $E_1, E_2, \ldots, E_n, \ldots \in \Sigma \Longrightarrow \cup_1^\infty E_n \in \Sigma$.

(We shall always assume $\emptyset \in \Sigma$ and $X \in \Sigma$.) Obviously, σ-algebras are also closed under countable intersections and set differences.

Examples.

1) For any set X, $\{\emptyset, X\}$, as well as the set of all subsets of X, is a σ-algebra on X.

2) The intersection of any family of σ-algebras (on the same set X) is again a σ-algebra on X.

3) Given any set S of subsets of a set X, the intersection of all σ-algebras on X which contain S is a σ-algebra on X, called the **σ-algebra generated by** S. It is the smallest σ-algebra on X which contains S. The case of interest to us is when X is a topological space, e.g. $\mathbb{R}^n$, and S is the set of all open subsets of X. The generated σ-algebra $\mathfrak{B}(X)$ in this case is called the **Borel algebra** of the space X.

Remark - Exercise. $\mathfrak{B}(\mathbb{R}^1)$ can also be generated by the set of all intervals $(-\infty, a)$, $a \in \mathbb{Q}$, or by the set of all closed bounded intervals etc.

Definition 1.2 *A measure* $\mathfrak{m}$ *on a σ-algebra Σ on a set X is a function* $\mathfrak{m} : X \to [0, \infty] \subset \mathbb{R} \cup \{\infty\}$ *(with $m(\emptyset) = 0$) which is* **countably additive***: if (E_n) is a sequence of pairwise disjoint sets in Σ, then*

$$\mathfrak{m}(\bigcup_n E_n) = \sum_n \mathfrak{m}(E_n).$$

Examples.

1) For any set X and any $x_0 \in X$, define for any $E \subset X$: $\delta(E) = 0$ if $x_0 \notin E$, $\delta(E) = 1$ if $x_0 \in E$. Then δ is a measure on the set of *all* subsets of X. It is called the **Dirac measure** on X at x_0.

2) For any set X and any $E \subset X$, define $\nu(E) =$ cardinality of E if E is finite, $\nu(E) = \infty$ otherwise. This is the so-called **counting measure** on X.

Interesting measures (on σ-algebras) are not easy to construct. Lebesgue realised that it was desirable and possible to have measures on σ-algebras, and that it is then possible to have a very satisfactory theory of integration. More specifically, he constructed a measure λ on a σ-algebra $\mathfrak{L}(\mathbb{R}^n)$ on $\mathbb{R}^n$ which is slightly bigger than the Borel algebra $\mathfrak{B}(\mathbb{R}^n)$, which assigns to each rectangular block (with sides parallel to the axes) its usual volume, and developed a beautiful and powerful theory of integration using this measure. In this course, we shall study some parts of Lebesgue's theory, and give some applications.

Construction of Lebesgue Measure.

By a **rectangle** in $\mathbb{R}^n$, we mean a subset of $\mathbb{R}^n$ of the form

$$R = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i, 1 \leq i \leq n\},$$

where $a_i, b_i \in \mathbb{R}$ and $a_i \leq b_i$, $1 \leq i \leq n$. Its volume $\lambda(R)$ is of course $\prod_{i=1}^n (b_i - a_i)$. Two rectangles are said to be **almost disjoint** if their interiors are disjoint. Observe that the intersection of two rectangles is again a rectangle, and that the union of two rectangles is a finite almost disjoint union of (possibly smaller) rectangles - this can be seen by subdividing each rectangle into smaller ones if necessary. Hence we can define the volume of a finite union P of rectangles - which we shall call a **polyhedron** - in the obvious way: write P as a union $\cup R_i$ of pairwise almost disjoint rectangles, and set $\lambda(P) = \sum \lambda(R_i)$. It is easy to check that this is well-defined, and that

$$\lambda(P \cup Q) \leq \lambda(P) + \lambda(Q)$$

for any two polyhedrons P, Q, with equality if they are almost disjoint.

Definition 1.3 *For any open U in $\mathbb{R}^n$,*

$$\lambda(U) = \sup\{\lambda(P) : P \subset U \text{ a polyhedron}\}.$$

Remark. Since every open set in $\mathbb{R}^n$ is an increasing union of polyhedra, this definition is forced by countable additivity.

Proposition 1.4 *For any family of open sets (U_α), $\lambda(\cup U_\alpha) \leq \sum \lambda(U_\alpha)$, with equality if the U_α are pairwise disjoint.*

Proof: Exercise — note that any polyhedron $P \subset \cup U_\alpha$ is the almost disjoint union of rectangles, each of which is contained in some U_α, by the Lebesgue Number Theorem. $\square$

Remark. Observe that $\lambda(U) < \infty$ for a bounded open U. Since any compact K can be written as $U \setminus (U \setminus K)$ with $U \supset K$ bounded open, $\lambda(K)$ is already determined (by additivity) for compact, hence closed, subsets of $\mathbb{R}^n$. But we shall proceed slightly differently for convenience of proofs.

Definition 1.5 *For any $E \subset \mathbb{R}^n$, the* **outer (Lebesgue) measure** *of E is*

$$\lambda^*(E) := \{\inf \lambda(U) : U \supset E \text{ open }\}.$$

Proposition 1.6 *λ^* is* **countably sub-additive**, *i.e. for any sequence (E_i) of subsets of $\mathbb{R}^n$, $\lambda^*(\cup E_i) \leq \sum \lambda^*(E_i)$.*

Proof: Follows from the analogous proposition for open sets. $\square$

Remark. λ^* is **not** additive on **all** subsets of $\mathbb{R}^n$, i.e. there exist disjoint subsets of $\mathbb{R}^n$, say E, F, such that $\lambda^*(E \cup F) < \lambda^*(E) + \lambda^*(F)$. However, it is not easy to find such examples.

Proposition 1.7 *Suppose (E_i) is a sequence of pairwise disjoint* **closed** *sets. Then*

$$\lambda^*(\cup E_i) = \sum \lambda^*(E_i).$$

Proof: Because λ^* is countably sub-additive, we need only prove $\lambda^*(\cup E_i) \geq \sum_{i=1}^{k} \lambda^*(E_i)$ for every k. But the E_i are **closed** and pairwise disjoint, hence their exist disjoint **open** sets U_i such that $E_i \subset U_i$ ($\mathbb{R}^n$ is a **normal** topological space!). Hence $\sum_{i=1}^{k} \lambda^*(E_i) = \lambda^*(\cup_{i=1}^{k} E_i)$, so we are done. $\square$ We can now identify a σ-algebra $\mathfrak{L} = \mathfrak{L}(\mathbb{R}^n) \supset \mathfrak{B}(\mathbb{R}^n)$ on which λ^* is (countably) additive.

Definition 1.8 *A subset E of $\mathbb{R}^n$ is* **Lebesgue measurable** *if, for every $\varepsilon > 0$, there exist F closed and U open such that $F \subset E \subset U$ and $\lambda(U \setminus F) < \varepsilon$. In this case the* **Lebesgue measure** *$\lambda(E)$ of E is $\lambda^*(E)$. The set of all Lebesgue measurable subsets of $\mathbb{R}^n$ is denoted by $\mathfrak{L} = \mathfrak{L}(\mathbb{R}^n)$.*

Examples.

1) Any $E \subset \mathbb{R}^n$ with $\lambda^*(E) = 0$ is Lebesgue measurable. Such sets are called **null sets**, and play a very important role (as exceptional sets) in many problems of analysis and geometry. A singleton is a null set, and a countable union of null sets is a null set. A null set has empty interior, but there exist compact sets of positive measure whose interiors are empty. The real axis in $\mathbb{R}^2$ (i.e. the X-axis) is a null set.

2) Any (open or closed) rectangle is measurable, and its measure is its usual volume.

3) Non-measurable sets are not easy to find. It can be shown that every measurable susbset of $\mathbb{R}^n$ of positive measure contains non-measurable sets.

4) If $\lambda^*(E) < \infty$, then $E \in L$ if and only if $\lambda^*(E) = \sup\{\lambda^*(F) : F \subset E \text{ closed}\}$. This is because, if $F \subset U$ with F closed and U open, then $\lambda(U) = \lambda(U \setminus F) + \lambda^*(F)$, as is easily verified from the definitions.

We can now prove the main result we were aiming for:

Theorem 1.9 *$\mathfrak{L}$ is a σ-algebra containing $\mathfrak{B}(\mathbb{R}^n)$, and $\lambda = \lambda^*|\mathfrak{L}$ is a measure on E.*

Proof: That $\mathfrak{L}$ is an **algebra**, i.e. closed under complementation and finite unions, is almost obvious. To prove (finite) additivity, suppose $E_1, E_2 \in \mathfrak{L}$ are disjoint. If $\lambda(E_1)$ or $\lambda(E_2) = \infty$, there is nothing to prove, so we assume $\lambda(E_1), \lambda(E_2) < \infty$. Then $\lambda(E_1 \cup E_2) < \infty$. If $F_i \subset E_i$ is closed, then

$$\begin{aligned} \lambda(E_1 \cup E_2) &\geq \lambda^*(F_1 \cup F_2) \\ &= \lambda^*(F_1) + \lambda^*(F_2) \text{ since } F_1 \cap F_2 = \emptyset \end{aligned}$$

Since the $F_i \subset E_i$ are arbitrary closed, we get $\lambda(E_1 \cup E_2) \geq \lambda(E_1) + \lambda(E_2)$ as desired. It remains only to prove (since $\mathfrak{L}$ is already an algebra), that $\mathfrak{L}$

is closed under countable disjoint unions. Countable additivity will then follow from finite additivity and countable sub-additivity. Thus, let (E_i) be a disjoint sequence in $\mathfrak{L}$. Given $\varepsilon > 0$, choose (F_i) and (U_i), closed and open respectively, $F_i \subset E_i \subset U_i$, $\lambda(U_i \setminus F_i) < \dfrac{\varepsilon}{2^{i+1}}$. The difficulty is of course that $\cup F_i$ may not be closed. But if $\sum \lambda(E_i) < \infty$ (hence $\sum \lambda(U_i) < \infty$), it is clear that $\lambda(\cup_1^\infty U_i \setminus \cup_1^k F_i) < \varepsilon$ if k is large enough, hence $\cup_i E_i \in \mathfrak{L}$. Now write $\mathbb{R}^n = \cup_1^\infty C_m$, where C_m are the unit cubes with points of $\mathbb{Z}^n$ as vertices. Then, by the above case, each $E'_m := \cup_i (C_m \cap E_i) \in \mathfrak{L}$. But note now that, if $F_m \subset C_m$ is closed for each m, then $\cup F_m$ is closed, because (C_m) is a **locally finite** family. Hence we can conclude that $\cup_m E'_m (= \cup_i E_i) \in \mathfrak{L}$ by the usual argument. Finally, $\mathfrak{L} \supset \mathfrak{B}(\mathbb{R}^n)$, since rectangles lie in $\mathfrak{L}$, and open sets are countable unions of rectangles. Proof of the theorem is complete. $\square$

We summarise:

Theorem 1.10 *A subset E of $\mathbb{R}^n$ is Lebesgue measurable if and only there is an increasing sequence of closed (or even compact) subsets (F_n) of E and a null set N such that $E = N \cup (\cup F_n)$, or equivalently, if and only if there is a decreasing sequence (U_n) of open sets containing E and a null set N' such that $E \cup N' = \cap U_n$; $\lambda(E) = \lim \lambda(F_n) = \lim \lambda(U_n)$.*

Proof: The sufficiency of the conditions is clear. Now suppose $E \in \mathfrak{L}$. For each $n \geq 1$, choose $F_n \subset E \subset U_n$, F_n closed and U_n open, with $\lambda(U_n \setminus F_n) < \frac{1}{n}$. Then $N := E \setminus \cup F_n$ and $N' := (\cap U_n) \setminus E$ are both subsets of $\cap (U_n \setminus F_n)$, hence null sets. $\square$

Remark. In the above theorem. the null sets N, N' **cannot be dropped**, e.g. $\mathbb{Q} \subset \mathbb{R}$ is not a G_δ (countable intersection of open sets). But we do have the remarkable fact that **every** Borel set in $\mathbb{R}^n$ is a G_δ "*upto a null set*"; similarly, it is **also** an F_σ (i.e. a countable union of closed sets) upto a null set.

2 Two Important Examples

Proposition 2.1 *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable with $\lambda(E) > 0$. Then the "vector difference"*

$$E - E = \{u - v : u, v \in E\}$$

contains a neighbourhood of the origin in $\mathbb{R}^n$.

Proof: By assumption, $E \supset K$ compact with $\lambda(K) := \lambda^*(K) > 0$. (Prove this!).

Then there exists $U \supset K$ open with $\lambda(U) < 2\lambda(K)$ say. We claim that

$K - K$ contains the ball B of radius $d(K, \mathbb{R}^n \setminus U)$ around the origin otherwise, we would

have $x \in B$ such that K and $K + \{x\}$ are disjoint. But we also have

$K + \{x\} \subset U$, contradicting $\lambda(U) < 2\lambda(K)$. Note that we have used the

translation invariance of $\mathfrak{L}$ and λ: if $E \in \mathfrak{L}$ and $x \in \mathbb{R}^n$, then

$E + \{x\} \in \mathfrak{L}$, and $\lambda(E + \{x\}) = \lambda(E)$. This property of $\mathfrak{L}$ and

λ follows immediately from the translation invariance of rectangles and their volumes.

For the second example, we fix any $E \in \mathfrak{L}$, and define, for any $x \in \mathbb{R}^n$:

$$D^u_E(x)(= D^u(x)) := \limsup_{\varepsilon \longrightarrow 0} \lambda(C(x, \varepsilon) \cap E)/\lambda(C(x, \varepsilon)),$$

where $C(x, \varepsilon)$ denotes the cube of edge 2ε with center x. For obvious reasons, $D^u(x)$ is called **upper density** of E at x. The **lower density** $D^l(x)$ is defined as the corresponding $\liminf$, and clearly

$$0 \leq D^l(x) \leq D^u(x) \leq 1.$$

If $D^l(x) = D^u(x)$, the ratio $\lambda(C(x, \varepsilon) \cap E)/\lambda(C(x, \varepsilon))$ converges to the common value, which is then called the **density** of E at x and denoted by $D(x) = D_E(x)$. Lebesgue's remarkable result is now as follows:

Theorem 2.2 *There exists a null set* N *in* $\mathbb{R}^n$ *such that, for all* $x \notin N$, $D_E(x) = D(x)$ *exists, and* $= 1$ *if* $x \in E \setminus N$ *and* $= 0$ *if* $x \notin E \cup N$.

Proof: If $E' := \mathbb{R}^n \setminus F$, it is clear that $D_E(x)$ exists if and only if $D_{E'}(x)$ exists, and then $D_E(x) + D_{E'}(x) = 1$. Thus it is enough to prove that

$$B := \{x \in E' : D^u_E(x) > 0\}$$

is a null set (verify this!). Hence it is sufficient (and neccessary) to prove that $B_k := \{x \in E' : D^u_E(x) > \frac{1}{k}\}$ is a null set for each integer $k \geq 1$ (because $B = \cup B_k$). Fix any $k \geq 1$. *Suppose we know that* B_k *is measurable.* Then B_k is a null set if and only if $\lambda(K) = 0$ for every compact $K \subset B_k$ (this is *false* for non-measurable sets). Thus, let $K \subset B_k$ be compact, and let $\varepsilon > 0$

be arbitrary. There exists $U \supset K$ open such that $\lambda(U) < \lambda(K) + \varepsilon$. Since $K \subset B_k$, there exists for each $x \in K$ a cube $C_x \subset U$ with center x such that

$$\lambda(C_x \cap E) > \lambda(C_x)/k.$$

Finitely many of the C_x, say $C_1, \ldots, C_s$ cover K. The C_l may not be disjoint, but there exists $J \subset \{1, 2, \ldots, s\}$ such that the C_j, $j \in J$ are disjoint and

$$\lambda(\cup_{j \in J} C_j) \geq 3^{-n} \lambda(\cup_1^s C_j).$$

Assuming this, we have

$$\begin{aligned} \varepsilon &> \lambda(U \setminus K) \geq \lambda((U \setminus K) \cap E) = \lambda(U \cap E) \\ &\geq \sum_{j \in J} \lambda(C_j \cap E) > \frac{1}{k} \sum_{j \in J} \lambda(C_j) \\ &\geq \frac{3^{-n}}{k} \lambda(\cup_{j=1}^s C_j) \geq \frac{3^{-n}}{k} \lambda(K). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\lambda(K) = 0$ as asserted. Finally, we claim that, for every $r > 0$, the set

$$S_r := \{x \in \mathbb{R}^n : D^u(x) > r\}$$

is measurable. Indeed, from the definition of lim sup, we see that $S_r = \cap_1^\infty T_l$, where

$$T_l := \cup_{0 < \varepsilon < 1/l} \{x \in \mathbb{R}^n : \lambda(C(x, \varepsilon) \cap E) > r\lambda(C(x, \varepsilon))\}.$$

Now, for each $\varepsilon > 0$, $\lambda(C(x, \varepsilon) \cap E)$ is a continous function of x, hence T_l is an open set, and we are done. ☐