Commutative Algebra

S. A. Katre[1]
email: sakatre@math.unipune.ernet.in

Problem Set-2

1. Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

2. A local ring contains no idempotent $\neq 0, 1$.

3. Show that the condition in *Prime Avoidance Lemma* cannot be improved.

   Example: Show that if $k = \mathbb{Z}/2\mathbb{Z}$, then the ideal $(x, y) \subset \dfrac{k[x, y]}{(x, y)^2}$ is the union of three properly smaller ideals.

4. The prime spectrum of a ring: (pg. 12: Ati-Mac)
   Let A be a ring and X = set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that

   (a) If $\mathfrak{a}$ is the ideal generated by E, then
   $$V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a})).$$
   (b) $V((0)) = X, \quad V((1)) = \phi$
   (c) If E_i is any family of subsets of A, then
   $$V(\bigcup E_i)_{i \in I} = \bigcap_{i \in I} V(E_i).$$
   (d) $V(\mathfrak{a} \bigcap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \bigcup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

   These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski Topology. *The topological space X is called the* Prime Spectrum *of A, and is written $Spec(A)$.*

---

[1]Problem set prepared for AFS-II, June 2006 held at Bhaskaracharya Pratishthana and Dept. of Mathematics, University of Pune, Pune

5. For each $f \in A$, let X_f = complement of $V(f)$ in $X = Spec(A)$. The sets X_f are open. Show that they form a basis of open sets for the Zariski Topology and that:

   (i) $X_f \cap X_g = X_{fg}$

   (ii) $X_f = \phi \Leftrightarrow f$ is *nilpotent*.

   (iii) $X_f = X \Leftrightarrow f$ is a *unit*.

   (iv) $X_f = X_g \Leftrightarrow r((f)) = r((g))$.

   (v) X is quasi- compact (i.e. every open covering of X has a finite sub-covering).

   (vi) More generally, each X_f is quasi-compact.

   (vii) An open subset of X is quasi-compact if and only if it is a finite union of the sets X_f.

6. Let $\phi : A \to B$ be a ring homomorphism.
   Let $X = Spec(A)$, $Y = Spec(B)$. If $\mathfrak{q} \in Y$, $\phi^{-1}(\mathfrak{q})$ is a prime ideal of A. Hence, ϕ induces a mapping $\phi^* : Y \to X$. Show that:

   (i) $f \epsilon A$, then $(\phi^*)^{-1}(X_f) = Y_{\phi(f)}$; hence ϕ^* is continuous.

   (ii) Let $\psi : B \to C$ be another ring homomorphism. Then,
   $$(\psi \circ \phi)^* = \phi^* \circ \psi^*.$$

7. If $\mathfrak{b}$ is an ideal in a ring A, the annhilator of $\mathfrak{b}$ is $(0 : \mathfrak{b}) := Ann(\mathfrak{b})$. Check that $Ann(\mathfrak{b})$ is an ideal of A.