Commutative Algebra

S. A. Katre[1]
email: sakatre@math.unipune.ernet.in

Problem Set-1

1. Show that the *units* of $\mathbb{Z}[i]$ are $\pm 1, \pm i$. Show that the ideal (2) in $\mathbb{Z}[i]$ is not a prime ideal, but is the square of a prime ideal.

2. Show that the rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[2^{1/3}]$ have infinitely many units.

3. If u is a *unit* and x is *nilpotent*, show that $u + x$ is a *unit*.

4. Give an example of a ring in which

   (a) *Jacobson* radical = *Nilradical*

   (b) *Jacobson* radical $\neq$ *Nilradical*

5. A is a ring. If $f(x) = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ is a *unit* and $g(x) = b_0 + b_1x + \cdots + b_mx^m \in A[x]$ is its inverse, then prove that

   $$a_nb_m = 0, a_n^2b_{m-1} = 0, \ldots, a_n^mb_1 = 0, a_n^{m+1} = 0;$$

   thus a_n is *nilpotent*.

   Deduce that a_0 is a *unit* and $a_1, a_2, \ldots, a_n$ are *nilpotent*.

   Show that in $A[x]$, *Jacobson* radical = *Nilradical*.

6. If $m_1, m_2, \ldots, m_r$ are coprime in pairs. Show that the system of congruences

   $$x \equiv x_i \pmod{m_i}, \quad 1 \leq i \leq r$$

   has an integer solution.

   (*Hint*: Get y_i such that $y_i \equiv 1 \pmod{m_i}$ and $y_i \equiv 0 \pmod{m_j}, j \neq i$.)

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