Exercises on Differentiable Manifolds

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1)* Let V be a compact non-orientable C^∞ manifold without boundary of dimension n. Show that there is no C^∞ embedding $V \subset \mathbb{R}^{n+1}$. In particular, a compact non-orientable surface cannot be differentiably embedded in $\mathbb{R}^3$.

(Hint: Use the following two facts.)

(a) $H_{n-1}(V, \mathbb{Z})$ contains $\mathbb{Z}/_{(2)}$ as a subgroup.

(b) Use the long exact cohomology sequence of the pair $(S^{n+1}V)$ and Poincare - Lefschetz - Alexander duality $H^i(S^{n+1}, V) \approx H_{n+1-i}(S^{n+1} - V)$.

2) Using the observation that a torus T^2 is homeomorphic to a surface of revolution in $\mathbb{R}^3$, find a polynomial equation for a suitable embedding $T^2 \subset \mathbb{R}^3$.

3) Let $U \subset \mathbb{R}^2$ be a non-empty open set and $f : U \to \mathbb{R}^2$ a C^∞ map. If $\frac{J(f_1,f_2)}{(x,y)} \equiv 0$ on U then for any compact $K \subset U$, there is a non-zero C^∞ function $\Phi(T_1, T_2)$ defined in an open set in $\mathbb{R}^2$ containing $f(K)$ such that $\Phi(f_1, f_2) \equiv 0$ on K and Φ is not identically 0 on any non-empty open subset.

(Hint: $f(K)$ has measure 0 by Sard's theorem and it is compact).

4) Give an example of a C^∞ map $\pi : \tilde{M} \to M$ between C^∞ manifolds of same dimension such that π is a submersion and also a surjection such that π is not a covering map.

5) Let $N \subset M$ be a closed C^∞ submanifold of a C^∞ manifold. Show that any C^∞ function on N can be extended to a C^∞ function on M.

(Hint: Use partition of unity)

6) Let $N \subset M$ be as in Exercise (5) above. Show any C^∞ vector field on N can be extended to a C^∞ vector field on M.

7) Let V be a C^∞ manifold which is second countable. Show that the ring of C^∞ functions $C^\infty(V)$ has the property that any ring homomorphism $\varphi : C^\infty(V) \to \mathbb{R}$ is evaluation at some point $p \in V$, i.e. there is a point $p \in V$ such that $\varphi(f) = f(p)$ for any $f \in C^\infty(V)$.

8) Let $S^1 \subset \mathbb{R}^4$ be a C^∞ embedding. Show that there exists a hyperplane H in $\mathbb{R}^4$ such that the orthogonal projection of C to H is an embedding.

9) Let $F(X_1, \cdots X_n)$ be a homogeneous polynomial of degree $d \geq 1$ with real coefficients. Show that for any constant $c \neq 0$, the subset $\{F(x_1, \cdots x_n) = c\} \subset \mathbb{R}^n$ is a closed submanifold of $\mathbb{R}^n$. If c_1, c_2 are real numbers with $c_1 \cdot c_2 > 0$ show that $\{F = c_1\}$ and $\{F = c_2\}$ are diffeomorphic. Find examples when this is false if $c_1 \cdot c_2 < 0$.

10) Let $U, V \subset \mathbb{C}^n$ be open subsets and $f : U \to V$ a complex analytic map. If the tangent maps $T_{U,a} \to T_{V,f(a)}$ are isomorphisms for all $a \in U$ then show that the determinant of the Jacobian of the real map (thinking of U, V as open subsets of $\mathbb{R}^{2n}$) is > 0 for any $a \in U$.

    Use this to show that any non-singular complex analytic hypersurface $f(z_1, \cdots z_n) = 0$ in $\mathbb{C}^n$ is orientable.

11) Let V be a compact C^∞ manifold without boundary. Show that any non-constant C^∞ map $V \to \mathbb{R}$ has at least two critical values.

12) Let D^n be the compact unit disc in $\mathbb{R}^n$ with boundary S^{n-1}. Let $f : D^n \to D^n$ be a C^∞ map such that $f/_{S^{n-1}} =$ Identity. Show that f is onto.

13) Show that any 1-dimensional differentiable manifold is orientable.

14) * Let V be a C$^\infty$ manifold and let $A \subset V$ be a closed subset. Suppose that there is C$^\infty$ map $f : V \to V$ such that $f(V) \subset A$ and $f|_A = \textit{identity}$. Show that A is a C$^\infty$ manifold.

15) Show that any product of spheres $S^{n_1} \times S^{n_2} \cdots \times S^{n_m}$ can be embedded in $\mathbb{R}^{\Sigma_1^m n_i + 1}$.

16) Let $f : S^1 \to \mathbb{R}$ be a differentiable map. Show that for any regular value $t \in \mathbb{R}$ the inverse image $f^{-1}(t)$ has an even number of points.

17) Regarding S^1 as an equator of S^2 we obtain the real projective line $\mathbb{R}P^1$ as a differentiable submanifold of $\mathbb{R}P^2$. Show that $\mathbb{R}P^1$ cannot be the inverse image under a differentiable map $\mathbb{R}P^2 \to \mathbb{R}$ of a regular value.

18) Find an example of a connected, non-orientable manifold V with non-empty boundary such that the boundary is orientable.

19) Let V, W be (connected) C$^\infty$-manifolds such that $V \times W$ is orientable. Show that V, W are orientable. (The converse is easily seen to be true.)

20) Let f be a C^∞ function in an open neighborhood of the origin in $\mathbb{R}^n$ such that $f(0) = 0$ and all the partial derivatives of f at 0 are 0. Show that $f = \sum_{i=1}^r \varphi_i f_i$, where φ_i are C^∞ near the origin and f_i are all the homogeneous quadratic polynomials in $x_1, x_2, \ldots, x_n$.