Differentiable Manifolds

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§ **1.** We will begin with results from the theory of differentiable functions of several real variables.

§ **1.1.** Let $U \subset \mathbb{R}^n$ be a non-empty open subset. We denote by $x_1, \cdots, x_n$ global coordinates on $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is said to be infinitely differentiable, or C^∞, if all the partial derivates $\frac{\partial^{i_1+\cdots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$ exist (and then they are also continuous) at any point in U. If $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are non-empty open sets and
$f : U \to' V$ a map such that each coordinate function of $f = (f_1, \cdots, f_m)$ is a C^∞ function on U then we say that f is a differentiable or smooth map.

Remarks (1) More generally, we can define functions which are C^k on open sets in $\mathbb{R}^n$ and maps which are C^k. A function is C^k if all its derivatives upto order k exist and are continuous on U, etc.

(2) A C^∞ function may not be analytic. For example,

$$\text{let } f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Let $\{r_m\}$ be an ordering of all rational numbers. The function $g(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} . f(x - r_m)$ is C^∞ but nowhere analytic, i.e. g is not represented by its Taylor's series. This is because each term in the above series is analytic at each point except at r_m. Hence if g is analytic in some open set then it is analytic at any r_m in that open set, a contradiction since all but one functions in the serie s is analytic at r_m.

In fact, we will show below that there exist non-trivial C^∞ functions on $\mathbb{R}^n$ which are identically 0 outside a compact subset of $\mathbb{R}^n$ (C^∞ Urysohn's lemma).

Notation. Will use the following multi-index notation in future.

$\alpha = (\alpha_1, \cdots, \alpha_n), \alpha_i \geq 0$ being integers.
$x = (x_1, \cdots, x_n), x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.
$D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}, \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!, \mid \alpha \mid = \alpha_1 + \cdots + \alpha_n$.
$\parallel x \parallel = (\sum x_i^2)^{1/2}$.

With these notations we will state the important Taylor's formula.

§1.2. Taylor's theorem

Let $U \subset \mathbb{R}^n$ be an open set and let $f \in C^k(U)$. Suppose $x, y \in U$ are such that the closed line segment $[x, y]$ joining x to y is also contained in U. Then we have

$$f(x) = f(y) + \Sigma_{|\alpha|=1}^{k-1} \frac{D^\alpha f(y)}{\alpha!}(x-y)^\alpha + \Sigma_{|\alpha|=k} \frac{D^\alpha f(\xi)}{\alpha!}(x-y)^\alpha$$

where $\xi \in [x, y]$ is a suitable point.

This follows from the 1-variable Taylor's formula applied to the function $g(t) = f(y + t(x-y))$, where t belongs to an open neighborhood of $[0, 1]$ in $\mathbb{R}$.

Let R_k denote the last term in Taylor's formula. Then for all x such that $|x - y| \leq \delta$ we have $|R_k| \leq M\delta^k$, where M depends on k and the upper bound on all the partial derivatives of f of total order k in U.

§1.3. C^∞ Urysohn's lemma

Our aim is to prove the following useful result.

Proposition 1. Let $A \subset \mathbb{R}^n$ be a compact set and U an open set in $\mathbb{R}^n$ containing A. Then there is a C^∞ function $f : \mathbb{R}^n \to [0,1]$ such that $f(A) = 1$ and the set $\{p \in \mathbb{R}^n \mid f(p) > 0\}$ has compact closure contained in U.

Proof. Step 1. For any point $a \in \mathbb{R}^n$ and $\epsilon > 0$, let $B_\epsilon(a)$ be the open ball of radius ϵ with center a. Let $h(t)$ be defined on $\mathbb{R}$ by

$$h(t) = 0, t \leq 0, \text{ and } h(t) = e^{-1/t} \text{ for } t > 0.$$

Then h is a C^∞ function. We write (x) for $(x_1, \cdots, x_n) \in \mathbb{R}^n$.

Let $\overline{g}(x) = \frac{h(\epsilon - \sum x_i^2)}{h(\epsilon - \sum x_i^2) + h(\sum x_i^2 - \frac{1}{2}\epsilon)}$

Then $\overline{g}$ is well defined and C^∞. Clearly $\overline{g}(x) = 0$ if $\sum x_i^2 \geq \epsilon$ and for all other $(x), \overline{g}(x) > 0$. If $0 \leq \sum x_i^2 \leq \frac{1}{2}\epsilon$ then $\overline{g}(x) \equiv 1$. Thus $\overline{g}(x)$ is C^∞, vanishes outside $B_{\sqrt{\epsilon}}(0)$ and > 0 on $B_{\sqrt{\epsilon}}(0)$. Further, $\overline{g}(x) \equiv 1$ on $\overline{B}_{\sqrt{\epsilon/2}}(0)$. Now $g(x) = \overline{g}(x - a)$ has the analogous property in $B_{\sqrt{\epsilon}}(a)$, viz. $g(x) \equiv 0$ outside $B_{\sqrt{\epsilon}}(a), > 0$ on $B_{\sqrt{\epsilon}}(a)$ and identically 1 on $\overline{B}_{\sqrt{\epsilon/2}}(a)$.

Step 2. Since A is a compact, we can find $\in > 0$ and finitely many points $a_1, \cdots, a_k \in A$ such that $A \subset \bigcup_{i=1}^{k} B_{\sqrt{\epsilon}/2}(a_i)$ and $\bigcup_{1}^{k} \overline{B}_{\sqrt{\epsilon}}(a_i) \subset U$. Let g_i be defined on $B_{\sqrt{\epsilon}}(a_i)$ with the properties in Step 1. Let $f(x) = 1 - \pi_{i=1}^{k}(1 - g_i)$.

For any $x \in A$ at least one g_i has value 1 at x. Hence $f\mid_A \equiv 1$. Clearly, each $g_i \equiv 0$ outside $\bigcup_{1}^{k} B_{\sqrt{\epsilon}}(a_i)$. Hence f is identically zero outside the compact set $\bigcup_{1}^{k} \overline{B}_{\sqrt{\epsilon}}(a_i)$ which is contained in U.

Corollary. Let $(a) \in \mathbb{R}^n, U$ an open set containing (a) and f a C^∞ function $U \to \mathbb{R}$. Then there is an open neighbourhood V of $(a), V \subset U$ such that $f' = f\ |_V$ extends to a C^∞ function $\mathbb{R}^n \to \mathbb{R}$.

§ 1.4. Partition of Unity

Recall that for a continuous function f on an open subset $U \subset \mathbb{R}^n$ the support of f is the closure of the set $\{(x) \in \mathbb{R}^n \mid f(x) \neq 0\}$. A family of subset $\{E_i\}_{i\in I}$ of U is called *locally finite* if any $a \in U$ has an open neighbourhood which meets only finitely many E_i. A family of subsets $\{E'_j\}_{j\in J}$ is called a refinement of the family $\{E_i\}_{i\in I}$ if each $E'_j \subset E_{j_i}$ for some E_{j_i} depending on j.

A topological space X is *locally compact* if any point $x \in X$ has an open neighborhood whose closure is compact.

Recall the following result from General Topology.

Lemma 1. Let X be a locally compact, Hausdorff topological space which is a countable union of compact sets. Then X is a paracompact, i.e. any open covering $\{U_i\}$ of X has a locally finite refinement by open sets $\{V_j\}$ such that $\overline{V}_j \subset U_{i_j}, \{V_j\}$ is a covering of X and $\overline{V}_j$ is compact.

Theorem 1. Let $U \subset \mathbb{R}^n$ be an open subset and $U = \bigcup\limits_{i\in I} U_i$, where each U_i is open. Then there exists a family of C^∞ functions $\{\varphi_i\}_{i\in I}$ such that

(i) $0 \leq \varphi_i \leq 1$ and support $\varphi_i \subset U_i$

(ii) $\{\text{support } \varphi_i\}_{i\in I}$ is a locally finite family

(iii) $\sum\limits_{i\in I} \varphi_i(x) = 1$ for each $x \in U$.

Proof. By Lemma 1 there is a locally finite refinement $\{V_j\}_{j\in J}$ such that $\overline{V_i}$ is compact, $\overline{V}_i \subset U_i$ and $U = UV_i$. We can find an open covering $\{W_j\}_{j\in J}$

such that $\overline{W_j} \subset V_j$ for $j \in J$. By Proposition 1, there exists a C^∞ function ψ_j on $U, \psi_j(x) > 0$ on w_j, support $\psi_j \subset U_j$ and $\psi_j \geq 0$. Define $\varphi'_j = \psi_j / \sum_{k \in J} \psi_k$. This is well-defined by local finiteness of $\{W_j\}$ and since $\psi_k(x) > 0$ for each x for some k. Now $\sum_{j \in J} \varphi_j = 1$. Let $\tau : J \to I$ be any map such that $V_j \subset U_{\tau(j)}$. For each $i \in \ I$, let $J_i = \tau^{-1}(i)$. Define $\varphi_i = \sum_{j \in J_i} \varphi'_j$. Then $\sum \varphi_i = 1$, support $\varphi_i \subset U_i$ and $\{supp\ \varphi_i\}$ is a locally finite family.

Corollary. Let $U \subset \mathbb{R}^n$ be open, X a closed subset of U and V an open subset of U containing X. Then there exists a C^∞ function $\psi : U \to \mathbb{R}$ such that $\psi(x) = 1$ for $x \in X, \psi(x) = 0$ for $x \in V - V, 0 \leq \psi \leq 1$.

Proof. By the Theorem above, there exist C^∞ function $\varphi_1, \varphi_2 \geq 0$ such that supp $\varphi_1 \subset V$, supp $\varphi_2 \subset U - X$ and $\varphi_1 + \varphi_2 = 1$ on U. Let $\varphi = \varphi_1$.

Remark. Later we will construct a partition of unity on any second countable differentiable manifold.

§2. Inverse and implicit function theorems

§2.1 Inverse function theorem

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^n$ a C^∞ function, $f = (f_1, \cdots, f_n)$, such that for some $a \in U$, the determinant $\mid \frac{\partial f_i}{\partial x_j} \mid (a) \neq 0$. Then there is an open neighbourhood $a \in V \subset U$ such that $f \mid_V$ is a homeomorphism onto an open neighbourhood of $f(a)$.

Proof. We can assume that $a = 0, f(a) = 0$. By composing f with a suitable non-singular linear map of $\mathbb{R}^n$ into itself, we can also assume that the Taylor expansion of f_i upto terms of degree 3 has the form $f_i = x_i + \sum_{\sum \alpha_i = 2} \frac{D^\alpha f(0)}{\alpha!} x^\alpha + \sum_{|\alpha=3} \frac{D^\alpha f(\xi)}{\alpha!}$. Here, Taylor's Theorem, ξ is a suitable point on the segment joining 0 and x. It is easy to see that for suitable real numbers $r_1 > 0, \cdots, r_n > 0$, letting $g = f(x) - x$ (in the multiindex notation), $| x_i |, | y_i | < r_i$ implies $| g(x) - g(y) | \leq 1/2 | x - y |$. Hence $f(x) - f(y) = g(x) - g(y) + x - y$ and for such x, y we have $| f(x) - f(y) | \geq \frac{1}{2} | x - y |$. This shows that f is injective on $W = \{x \mid | x_i | < r_i\}$. Let $V = \{y \in \mathbb{R}^n \mid | y_i | \leq \frac{1}{2} r_i\}$ and $B = W \cap f^{-1}(V)$. We will construct the inverse function inductively.

Let $\varphi_0 : V \to W$ be defined by $\varphi_0(y) = 0$ and by induction $\varphi_k(y) = y - g(\varphi_{k-1}(y))$. Using the observation above that $| g(x) - g(x') | \leq 1/2 | x - x' |$ on $\overline{W}$, we verify by induction that $\varphi_k(y) \in W$ for each k and further

$$| \varphi_k(y) - \varphi_{k-1}(y) | = | g(\varphi_{k-1}(y)) - g(\varphi_{k-2}(y)) | \leq \frac{r}{2^k} (r = (\sum r_i^2)^{1/2}).$$

Let $\varphi(y) = lim \varphi_k(y)$ as $k \to \infty$.

Since $g(x) = f(x) - x$, we deduce that $f(\varphi(y)) = y$. Since $f \;|_W$ is injective, φ is the inverse of f. Each φ_k is continuous and the convergence $\varphi_k \to \varphi$ is uniform. Hence φ is continuous.

Remark. The analogous statement and proof is valid for a holomorphic function f from an open set $U \subset \mathbb{C}^n$ to $\mathbb{C}^n$.

§ 2.2. Implicit function theorem

Let $U_1 \subset \mathbb{R}^m, U_2 \subset \mathbb{R}^n$ be open sets and $f : U_1 \times U_2 \to \mathbb{R}^n$ a C^∞ function. Let x, y be global coordinates on $\mathbb{R}^m, \mathbb{R}^n$ respectively. Suppose that for some $(a, b) \in U_1 \times U_2, f(a, b) = 0$ and the matrix $\left(\frac{\partial f_i}{\partial y_j}\right)(a, b)$ has rank n. Then there is an open neighbourhood $U \times V$ of (a, b) such that for any $x \in U$, there is a unique $y = y(x) \in V$ such that $f(x, y(x)) = 0$. Further, the map $x \to y(x)$ is continuous.

Proof. We deduce this result from the Inverse function theorem. For convenience, we denote the coordinates in $\mathbb{R}^m$ and $\mathbb{R}^n$ by $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ respectively. Define $F : U_1 \times U_2 \to \mathbb{R}^{m+n}$ by $F(x, y) = (x, f(x, y))$. We check easily that the determinant $\begin{vmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{vmatrix} (a, b) \neq 0$. By the previous result, there is a neighbourhood $U' \times V$ of (a, b) and a neighbourhood W of $(a, 0)$ such that $F |_{U' \times V} \to W$ is a homeomorphism. Let $\varphi : W \to U' \times V$ be the continuous inverse of F. Let U be a neighbourhood of a such that $x \in U$ implies $(x, 0) \in W$. For $x \in U$, let $y(x)$ be the projection of $\varphi(x, 0)$ on V. Clearly if $y \in V$ is such that $f(x, y) = 0$ then $y = y(x)$. Then $y(x)$ is a continuous map with $f(x, y(x)) = 0$.

Remark. Analogous result is valid for holomorphic maps $\mathbb{C}^{m+n} \to \mathbb{C}^n$. In this case $y(x)$ is a holomorphic function of x.

§2.3. Differentiability of the implicit functions

We will not give details for the general case since it only involves more complicated notation and no other serious difficulties.

Assume that $f(x, y_1, \cdots, y_n)$ is C^2 in an open neighbourhood U of 0 in $\mathbb{R}^{n+1}$, $f(0) = 0$ and $\frac{\partial f}{\partial x}(0) \neq 0$. By Implicit function theorem, there is an open neighbourhood V of 0 in $\mathbb{R}^n$ and a continuous function $g : V \to \mathbb{R}$ such that $f(g(y_1, \cdots, y_n), y_1, y_2, \cdots, y_n) \equiv 0$ for $(y) \in V$. By Taylor's formula, we can find an open subset $U' \subset U$ and a constant $M > 0$ such that for any $(x, y_1, \cdots, y_n) \in U'$,

$$f(x, y_1, \cdots, y_n) = \frac{\partial f}{\partial x}(0)x + \frac{\partial f}{\partial y_1}(0)y_1 + \cdots + \frac{\partial f}{\partial y_n}(0)y_n + R,$$

where $| R |\leq M(x^2+y_1^2+\cdots+y_n^2)$ in U'. Hence if V is a sufficiently small neighbourhood of 0 in $\mathbb{R}^n$ then $(g(y),y_1,\cdots,y_n)\in U'$ and we have

$$f(g(y),y_1,\cdots,y_n)=0=\frac{\partial f}{\partial x}(0).g(y)+\frac{\partial f}{\partial y_1}(0)y_1+\cdots+\frac{\partial f}{\partial y_n}(0)y_n+R.$$

If we vary only y_i and keep the other $y_j=0$, we set

$$0=\frac{\partial f}{\partial x}(0).g(0,\cdots,y_i,\cdots,0)+\frac{\partial f}{\partial y_i}(0)y_i+R.$$

¿From this we see that $\lim_{y_i\to 0}\frac{g(0,\cdots,y_i,0,\cdots,0)}{y_i}\to-\frac{\frac{\partial f}{\partial y_i}(0)}{\frac{\partial f}{\partial x}(0)}.$ This shows that g is differentiable (of order at least 1).

Since the Inverse function theorem is a special case of the Implicit function theorem the same discussion applies for differentiability of the inverse function.

§2.4. An example

Let $f(x_1,\cdots,x_n)=x_1^{a_1}+x_2^{a_2}+\cdots+x_n^{a_n}-1$, where a_i are integers ≥ 1. The only point in $\mathbb{R}^n$ where all the partials $\frac{\partial f}{\partial x_i}$ are 0 is the origin. Hence at any point p on the hypersurface $\{f=0\}$ in $\mathbb{R}^n$, $\frac{\partial f}{\partial x_i}(p)\neq 0$ for at least one i. Then in a neighbourhood of p, there exists a differentiable function, $g(x_1,\cdots,\widehat{x}_i,\cdots,x_n)$ such that $x_1^{a_1}+\cdots+g^{a_i}+\cdots+x_n^{a_n}-1\equiv 0$. Hence the map $(x_1,\cdots,\widehat{x}_i,\cdots,x_n)\to(x_1,\cdots,g,\cdots,x_n)$ is a parametrization of an open subset of the hypersurface $\{f=0\}$.

Similar result is valid for common zeros of finitely many differentiable functions by making use of the Implicit function theorem.

§2.5. Rank Theorem

Definition. Let $U,V\subset\mathbb{R}^n$ be open sets. A surjective homeomorphism $f:U\to V$ such that f and f^{-1} are both C^∞ is called a diffeomorphism.

Theorem. Let $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ be open sets and let $f = (f_1, \cdots, f_n)$ be a C^∞ map $U \to V$. Let $x_1, \cdots, x_m$ be the coordinates on $\mathbb{R}^m$. Suppose that the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)(x)$ has rank r for each $x \in \mathbb{R}^m$. Then by composing f with suitable diffeomorphisms of U and V, we can assume that f has the form $(x_1, \cdots, x_m) \to (x_1, x_2, \cdots, x_r, 0, \cdots, 0)$.

Proof. This is an application of the Implicit function theorem. Assume that $f(0) = 0$. We will indicate the argument only for the case when $m \geq n$ and $r = n$. Assume for simplicity that $\left|\frac{\partial f_i}{\partial x_j}\right|_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}(0) \neq 0$. Introduce new variables $y_1, \cdots, y_r$ and $g_i = f_i - y_i$ for $1 \leq i \leq n$. Then g_i are differentiable. By the Implicit function theorem, $\exists C^\infty$ functions

$$h_1(x_{n+1}, \cdots, x_m, y_1, \cdots, y_n), \cdots, h_n(x_{n+1}, \cdots, x_m, y_1, \cdots, y_n)$$

in an open neighbourhood of 0 in $\mathbb{R}^m$ such that

$$f_i(h_1, \cdots, h_1, x_{n+1}, \cdots, x_m) = y_i \text{ for } i = 1, 2, \cdots, n.$$

§ 3. Differentiable manifolds

3.1. Let X be a Hausdorff topological space. Suppose that any $x \in X$ has an open neighbourhood which is homeomorphic to an open ball in some $\mathbb{R}^n$. Then X is called a topological manifold. If X is connected then it is easy to see that the integer n is independent of $x \in X$. In this case we say that X is a topological manifold of dimension n. Using the theorem about invariance of dimension we can see that if X is connected then the integer n is well-defined. Now we will assume that X is connected.

Suppose now that X has an open covering $\{U_i\}$ such that there is a homeomorphism $f_j : U_i \to B_i \subset \mathbb{R}^n$, where B_i is the unit open ball in $\mathbb{R}^n$. Assume that whenever $U_i \cap U_j \neq \phi$ for some $i \neq j$, the natural map $f_i(U_i \cap U_j) \xrightarrow{f_{ij}} f_j(U_i \cap U_j)$ is a C^∞ map. In this case we say that X is a C^∞

manifold of dimension n. The function f_i is called a *coordinate charts* and f_{ij} the transition functions. Using the coordinate charts we can introduce many notions from the study of C^∞ functions an open sets in $\mathbb{R}^n$. For example

(1) Let $V \subset X$ be a non-empty open subset. Let $g : V \to \mathbb{R}$ be a function. We say that g is C^∞ if for each i the map $g \circ \ f_i^{-1}(V \cap U_i) \to \mathbb{R}$ is C^∞.

(2) More generally, let X, Y be C^∞ manifolds and $g : X \to Y$ a map. Let $\{(U_i, f_i)\}, \{(V_j, h_j)\}$ denote coordinates charts for X, Y respectively. This gives natural maps. $f_i(U_i) \to h_j(g(U_i))$ for any i, j whenever $V_j \cap g(U_i) \neq \phi$. If for any such i, j these maps are C^∞ maps then we say that g is a C^∞ map.

Remark. H. Whitney has proved that any paracompact C^1 manifold carries a real analytic structure. This structure is unique. In view of this considering only C^∞ manifolds is not a loss of generality.

§3.2. Examples

1) Let $S^n \subset \mathbb{R}^{n+1}$ be the hypersurface $\{x_1^2 + \cdots + x_{n+1}^2 = 1\}$. We have seen earlier that at any point on S^n where $x_{n+1} \neq 0$, the functions $x_1, \cdots, x_n$ serve as local parameters, i.e. the map $(x_1, \cdots, x_{n+1}) \to (x_1, \cdots, x_n)$ is a homeomorphism in a small neighbourhood of such a point. Since x_{n+1} is a C^∞ function of $x_1, \cdots, x_n$ in such a neighbourhood we see easily that S^n is a C^∞ manifold.

2) Let X, Y be differentiable manifolds of dimension m, n and let $(U_i, f_i), (V_j, g_j)$ be coordinate charts for X, Y respectively. Then $U_i \times V_j \xrightarrow{(f_i, g_i)} B_i \times B_j$ is a homeomorphism and the transition functions are C^∞.

3) Let $\tau : S^n \to S^n$ be the diffeomorphism $\tau(x) = -x$. If (U, f) is a coordinate chart for the C^∞ structure on S^n then we take the quotient

space $\mathbb{R}P^n$ of S^n obtained by identifying x and $-x$ $\forall x \in S^n$. Let $\pi : S^n \to \mathbb{R}P^n$ be the quotient map. Then $(\pi(U), f \circ \pi^{-1})$ serve as coordinate charts for $\mathbb{R}P^n$.

4) $SL(n,\mathbb{R})$. Let $SL(n,\mathbb{R})$ be the set of $n \times n$ matrices with real entries and determinant 1. Write the entries as (X_{ij}). Then $SL(n,\mathbb{R})$ can be considered as a closed hypersurface in $\mathbb{R}^{n^2}$ defined by $\{D = 1\}$, where $D =\mid X_{ij} \mid$. Clearly at any point p on $\{D = 1\}$, at least one partial derivative $\frac{\partial D}{\partial X_{ij}}(p) \neq 0$. By implicit function theorem, near p the function $X_{ij} \mid_{\{P=1\}}$ is a C^∞ function of the remaining variables $X_{k\ell}$. Hence $SL(n,\mathbb{R})$ is a manifold of dimension $n^2 - 1$.

§3.3. Partitions of unity for C^∞ manifolds

An easy modification of the partition of unity result proved earlier for an open subset of $\mathbb{R}^n$ yields the following general result.

Theorem of Partition of unity for C^∞ manifolds

Given an open covering $\{U_i\}$ of a C^∞ manifold V which is second countable, there exists a family of C^∞ function $\{\varphi_i\}$ such that $\varphi_i \geq 0$, support $\varphi_i \subset U_i$ such that the family $\{supp\ \varphi_i\}$ is locally finite and $\sum \varphi_i(x) = 1$ for each $x \in V$.

Proof. We have only to use the fact that if (U, φ) is a coordinate chart and $K \subset U$ is compact, then there a C^∞ function η on V with compact support $\subset U$ such that $\eta(x) > 0$ for $x \in K$. This was already proved before.

Corollary. If F is a closed subset of $V, F \subset U$ an open set, then there exists a C^∞ function φ on V such that

$$\begin{aligned} \varphi(x) &= 1 \quad \text{if } x \in F \\ &= 0 \quad \text{if } x \in X - U. \end{aligned}$$

§ 4. Tangent and cotangent vectors

§ **4.1** For an open set $U \subset \mathbb{R}^n$ and a differentiable map $f : (-1,1) \to U$, the image is a curve passing through $f(0)$. Let the defining functions be denoted by $x_1(t), \cdots, x_n(t)$ where $t \in (-1,1)$. The vector $(\frac{dx_1}{dt}(0), \cdots, \frac{dx_n}{dt}(0))$ is the tangent direction to this curve at the point $f(0)$. We will now define the tangent space to a manifold M which does not depend on embedding M in $\mathbb{R}^n$.

Definition Let M be a C^∞ manifold and $a \in M$. Consider all ordered pairs (f, U) where U is an open neighbourhood of a and f is a C^∞ map $U \to \mathbb{R}$. Define an equivalence relation on the set of ordered pairs as follows. $(f, U) \sim (f', U')$ if there is an open neighbourhood V of a such that $V \subset U \cap V'$ and $f |_V = f' |_V$. The equivalence classes are called germs of C^∞ functions at a, denoted by C_a^∞. We define a *tangent vector* L at a to be an $\mathbb{R}$-linear map $L : C_a^\infty \to \mathbb{R}$ such that for $f, g \in C_a^\infty, L(f \cdot g) = L(f) \cdot g(a) + f(a)L(g)$, i.e., L is a *derivation* of C_a^∞ into $\mathbb{R}$.

Given a germ (f, U), where (U, φ) gives a coordinate chart at a, the condition that $\frac{\partial f}{\partial x_1}(a) = 0 = \cdots = \frac{\partial f}{\partial x_n}(a)$ is well-defined. For any $f, g \in C_a^\infty, \varphi = f \cdot g - f(a)g - f \cdot g(a)$ has the property that (noting that $f(a), g(a)$ are constants) that $L(\varphi) = 0$. For any $r \in \mathbb{R}$, consider the constant functions $1, r \in C_a^\infty$. Now $L(r \cdot 1) = L(r) \cdot 1 + r.L(1)$. Hence $L(1) = 0$ and by linearity of $L, L(r) = 0$ for any $r \in \mathbb{R}$. More generally, let $f \in C_a^\infty$ be such that

$\frac{\partial f}{\partial x_i}(a) = 0$ for $1 \leq i \leq n$. We use the Taylor's theorem,

$$f(x) = f(a) + \frac{D^2 f(\xi)}{2!}(x-a)^2 (\text{multi} - \text{index notation!})$$

for some $\xi \in [a,x]$. Since f is C^∞ we can see that the terms in $\frac{D^2 f(\xi)}{2!}$ are again C^∞ functions of x. It is clear that $L(\frac{D^2 f}{2}(x-a)^2) = 0$. Thus, L is trivial on the subspace of functions in C_a^∞ whose first derivatives are 0 at a. Let $\mathcal{M}_a \subset C_a^\infty$ be the subspace of functions which vanish at a, we see that L acts naturally on $\mathcal{M}_a/\mathcal{M}_a^2$ and the vector space of all the tangent vectors at a is naturally isomorphic to the dual $Hom_{\mathbb{R}}(\mathcal{M}_a/\mathcal{M}_a^2, \mathbb{R})$. It is denoted by $T_a(M)$. Hence $dim_{\mathbb{R}} T_a(M) = dim\ R$.

If $U \subset \mathbb{R}^n$ is an open set and $a \in U$, we define the tangent spaces $(\frac{\partial}{\partial x_1})_a, \cdots, (\frac{\partial}{\partial x_n})_a$ by $(\frac{\partial}{\partial x_i})_a(f) = \frac{\partial f}{\partial x_i}(a)$ for $f \in C_a^\infty$. It is easy to verify that the n tangents $(\frac{\partial}{\partial x_i})_a$ are linearly independent, hence $T_a(U) = \displaystyle\sum_{i=1}^{\eta} \mathbb{R}(\frac{\partial}{\partial x_i})_a$. Now let (U, φ) be a coordinate chart of $a \in M$. For any $f \in C_a^\infty$, define $(\frac{\partial}{\partial x_i})_a(f) = \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(a))$ for $1 \leq i \leq n$. Here $x_1, \cdots x_n$ are coordinates on $\varphi(U) \subset \mathbb{R}^n$.

Definition The dual $Hom_{\mathbb{R}}(T_a(M), \mathbb{R})$ is called the space of differentials (or cotangent vectors, or co-vectors) at a. Clearly, if we define $(dx_i)_a \in Hom_{\mathbb{R}}(T_a(M), \mathbb{R})$ by $(dx_i)_a((\frac{\partial}{\partial x_j})_a) = \delta_{ij}$ then $(dx_1)_a, \cdots, (dx_n)_a$ form a basis of the space of differentials at a, which we denote by $T_a^*(M)$. For any $f \in C_a^\infty(U)$, define $(df)_a$ by $(df_a)((\frac{\partial}{\partial x_j})_a) = \frac{\partial f}{\partial x_j}(a)$. Here $U \subset \mathbb{R}^n$ is an open set. More generally, if (U, φ) is a coordinate chart at a and $f \in C_a^\infty$, we define $(df)_a$ by $(df)_a((\frac{\partial}{\partial x_i})_a) = \frac{\partial(f \circ \varphi^{-1})}{\partial(x_i)}(\varphi(a))$.

Again, for a differentiable map $\gamma : (-1, 1) \to U \subset \mathbb{R}^n$ and $f \in C_a^\infty(U)$, define $X(f) = \frac{d}{dt}(f \circ \gamma(t))(0)$. Then $X(f) = \displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot \frac{dx_i}{dt}$, where $\gamma(t) = (x_1(t), \cdots, x_n(t))$ and $a = \gamma(0)$. The vector $(\frac{dx_1}{dt}(0), \cdots, \frac{dx_n}{dt}(0))$ is the tangent to the parametrized curve in U defined by γ. This illustrates the relation between the intuitive idea of tangent to a curve and the abstract definition of a tangent vector at a. It is easy to see that given $X \in T_a(U)$, there is a parametrized curve $\gamma : (-1, 1) \to U$ such that the two meanings of $X(f)$ coincide for any $f \in C_a^\infty$.

§ 4.2. Tangent and cotangent maps.

Let V, W, be C^∞ manifolds of dimensions n and m respectively and $f : V \to W$ a C^∞ map. We define the maps. $f_* : T_a(V) \to T_{f(a)}(W)$ and $f^* : T^*_{f(a)}(W) \to T^*_a(V)$ by $f_*(X)(g) = X(g \circ f)$ and $(f^*(d\varphi))_a = (d(\varphi \circ f))_a$, where $g \in C^\infty_{f(a)}(W), d\varphi \in T^*_{f(a)}(W)$ and $X \in T_a(V)$. It is easy to verify that if g has all the first derivatives 0 at $f(a)$ then $g \circ f$ has all the first derivatives 0 at a.

§4.3. Tangent and cotangent bundles

Let V be a C^∞ manifold of dimension n. Let $T(V) = \coprod_{a \in V} T_a(V)$ be the disjoint union of tangent spaces at points in V. We will indicate how $T(V)$ is itself a C^∞ manifold of dimension $2n$.

Let $\{(U_i, \varphi_i)\}$ be coordinate charts covering V. Suppose $a \in U_i \cap U_j$. Let $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ be coordinates on U_i, U_j respectively. Any tangent vector X at a has the form $X = \sum_1^n a_i (\frac{\partial}{\partial x_i})_a$ and $X = \sum_1^n b_i (\frac{\partial}{\partial y_i})_a$. Using the transition function $U_i \cap U_j \to U_i \cap U_j$ let $y_j = h_j(x_1, \cdots, x_n)$ for $1 \le j \le n$. Now for any $f \in C_a^\infty, X(f) = \sum a_i \frac{\partial f}{\partial x_i} = \sum b_j \frac{\partial f}{\partial y_j}$. Hence $\sum_i a_i \{\sum_j \frac{\partial f}{\partial y_j} \cdot \frac{\partial h_j}{\partial x_i}\} = \sum b_j \frac{\partial f}{\partial y_j}$. This gives $\sum_i a_i \frac{\partial h_j}{\partial x_i} = b_j$ for $1 \le j \le n$, i.e. $b_j = \sum_i a_i \frac{\partial y_j}{\partial x_i}$. The vector $(b_1, \cdots, b_n) = (a_1, \cdots, a_n) \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots\cdots & \frac{\partial y_n}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial y_1}{\partial x_n} & \cdots\cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$

Now let E be the disjoint union $\coprod_i U_i \times \mathbb{R}^n$ and define the equivalence relation $((x_i), \nu) \sim ((y_j), w)$ if the point $(x_i) = (y_j) \in U_i \cap U_j$ and $(w_1, \cdots, w_n) = (\nu_1, \cdots, \nu_n) \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots\cdots & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_1}{\partial x_n} & \cdots\cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$

Let $\pi : E \to E/\sim$ be the natural map and $p' : E \to V$ be the continuous map $p'(x_i), \nu) \to (x_i)$. Then p' factors through $E/\sim$ and defines a continuous map $p : E/\sim \to V$. $E/\sim$ is called the tangent bundle of V. It is easy to show that $E/\sim$ is Hausdorff. Clearly, $p^{-1}(U_i) \approx U_i \times \mathbb{R}^n$.

In a similar fashion, we can construct the *cotangent bundle* of V, denoted by $T^*(V)$ or $\Omega^1(V)$.

$T_a^*(V)$ is generated by $(dx_1)_a, \cdots, (dx_n)_a$ in U_i and $(dy_1)_a, \cdots, (dy_n)_a$. In $U_i \cap U_j$, we have $(dy_i)_a = \sum \frac{\partial y_i}{\partial x_j}(dx_j)_a$. This give patching diffeomorphism $(U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ for $T^*(V)$. If we let $\wedge^p T^*(V) = \coprod \wedge^p T_a^*(V)$ for $1 \leq p \leq n$ and give a natural structure of a C^∞ manifold in a similar manner, we get a manifold of dimension $n + \begin{pmatrix} n \\ p \end{pmatrix}$. This can be called the bundle of differential p-forms.

§ 4.4. Vector fields and differential forms

Let $p : T(V) \to V$ be the natural projection A C^∞ vector field X on V is a C^∞ map $X : V \to T(V)$ such that $p \circ X =$ identity on V. On a coordinate chart $(U, \varphi); X_a = \sum h_i(a)(\frac{\partial}{\partial x_i})_a$ for any $a \in U$, where $a \to h_i(a)$ are C^∞ function on U.

Similarly, a C^∞ map $w : V \to \wedge^p T^*(V)$ such that $w(a) \in \wedge^p T_a^*(V)$ for $a \in V$ is called a C^∞ p-form. Again on (U, φ) as above

$$w_a = \sum_{i_1 < i_2 < \cdots < i_p} h_{i_1 \cdots i_p}(a)(dx_{i_1})_a \wedge \cdots \wedge (dx_{i_p})_a,$$

where $h_{i_1 i_2 \cdots i_p}$ are C^∞ functions on U.

Remark. $\wedge^p T_a^*(V)$ is the dual vector space of $\wedge^p T_a(V)$.

§4.5. Immersion and submersion

Given a C^∞ map $V \to W$ between C^∞ manifolds V, W respectively, we say that f has rank r at $a \in V$ if rank $f_* : T_a(V) \to T_{f(a)}(W)$ is r, i.e., the dimension of the image is r. Let $(a_1, \cdots, a_n) \in T_a(V)(i.e., X = \sum_{i=1}^{n} a_i (\frac{\partial}{\partial x_i})_a \in T_a(V))$, then $f_*(a_1, \cdots, a_n) = (b_1, \cdots, b_m)$, where $b_j = \sum_{i=1}^{n} a_i \frac{\partial f_j}{\partial x_i}(a)$. The rank of the map f_* is r if the matrix $(\frac{\partial f_i}{\partial x_j}(a))$ has rank r. If f_* is $1-1$ for each $a \in V$ the f is called an immersion and if f_* is surjective then f is called a submersion.

Remark If f is an immersion then $f(V)$ may not be a closed subset of W. But if f is proper then $f(V)$ is closed in W.

Suppose f is an immersion. Then by Rank Theorem, we can assume (after making local C^∞ diffeomorphisms in V and W) that $f_*(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, \cdots, 0)$.

§4.6. Sard's Theorem

Let $U, V \subset \mathbb{R}^n$ be open subsets and $f : U \to V$ a diffeomorphism. For any subset $S \subset U$ such that S has measure 0 the measure of $f(S)$ is also 0. Now let W be any C^∞, second countable manifold. Then the notion of a measure 0 subset $S \subset W$ is well-defined.

Definition Let V, W be C^∞ manifolds of dimensions n, m respectively and $f : V \to W$ a C^∞ map. A point $a \in V$ is called a *critical point* for f if rank

f_* at $a < m$. If $f^{-1}(w)$ does not contain any critical point then w is called a *regular value*. Otherwise, w is called a *critical value*.

Now we come to a fundamental result in differential topology.

Sard's Theorem. Let $f : V \to W$ be a C^∞ map between second countable manifolds. Then the image under f of the set of critical points in V has measure 0 in W.

Proof Since image under a C^∞ diffeomorphism of a measure 0 subset of an open subset of $\mathbb{R}^n$ again has measure 0, we need to consider only the following case. $f : U \to \mathbb{R}^p$ is a C^∞ map with U open in $\mathbb{R}^n$ and $p \geq 1$. Let C be the set of critical points; that is the set of all $x \in U$ with rank $df_x < p$. Then we must prove that $f(C)$ has measure 0 in $\mathbb{R}^p$. The set C is clearly closed. The proof is by induction on n. Let C_i denote the set of $x \in U$ such that all partial derivatives of f of order $\leq i$ vanish at x. Then $C \supset C_1 \supset C_2 \supset \cdots$.

Step 1 $f(C - C_1)$ has measure 0.

Proof If $p = 1$ then $C = C_1$, hence we can assume that $p \geq 2$. We need the following consequence of Fubini's theorem.

Lemma 2. A measurable set $A \subset \mathbb{R}^p = \mathbb{R}^1 \times \mathbb{R}^{p-1}$ has measure 0 if its intersection with each hyperplane $\{r\} \times \mathbb{R}^{p-1}$ is a set of measure 0 in $\mathbb{R}^{p-1}$.

We will show that for any $x \in C - C_1$, there is an open neighbourhood $V \subset \mathbb{R}^n$ such that $f(V \cap C)$ has measure 0. Since $C - C_1$ is covered by countably many of such neighbourhoods, $f(C - C_1)$ has measure 0.

Let $x \in C - C_1$. Suppose $\frac{\partial f_1}{\partial x_1}(x) \neq 0$. We can assume without loss of generality that in a suitable neighbourhood V of $x, f_1(x) = x_1$. Hence the map f has the form $(x_1, x_2, \cdots, x_n) \to (x_1, f_2, \cdots, f_p)$. If follows that a point in $\{t\} \times \mathbb{R}^{n-1}$ is critical for f iff it is critical for the map $\overline{f}(x_2, \cdots, x_n) =$

$(f_2(t, x_2, \cdots, x_n), \cdots, f_p)$. By induction on n, the set of critical values has measure 0 in $\{t\} \times \mathbb{R}^{p-1}$. Since C is closed, $f(C)$ is a countable union of compact subsets. Hence by Fubini's theorem $f(V \cap C)$ has measure 0.

Step 2 We will prove that for $i \geq 1, f(C_i - C_{i+1})$ has measure 0.

Let $x \in C_k - C_{k+1}$. We can assume that $\frac{\partial^{k+1} f_1}{\partial^{s_1} x_1 \partial^{s_2} x_2 \cdots \partial^{s_n} x_n}(x) \neq 0$. Assume that $s_1 > 0$. The kth derivative $w(x) = \frac{\partial^k f_1}{\partial^{(s_1-1)} x_1 \cdots \partial^{s_n} x_n}(x) = 0$ but $\frac{\partial^{k+1} f_1}{\partial^{s_1} x_1 \cdots \partial^{s_n} x_n}(x) \neq 0$. The map $(x_1, \cdots, x_n) \xrightarrow{h} (w, x_2, \cdots, x_n)$ is a diffeomorphism in an open neighbourhood V of x onto an open neighbourhood V' of 0 in $\mathbb{R}^n$. Consider $f \circ h^{-1} : V' \to \mathbb{R}^p$. Using an argument similar to step 1 we see that $f(C_k - C_{k+1})$ has measure 0.

Step 3. For large $k, f(C_k)$ has measure 0. Assume that $k > \frac{n}{p-1}$. Let $I^n \subset U$ be a cube of edge δ.

By Taylor's theorem and definition of $C_k, f(x+h) = f(x) + R(x,h)$, where $\| R(x,h) \| \leq C \| h \|^{k+1}$ for $x \in C_k \cap I^n, x+h \in I^n$. The constant C depends only on f and I^n. For any integer r, divide I^n into r^n cubes of edge $\frac{\delta}{r}$. Suppose $x \in I_1$ where I_1 is a cube in the subdivision. Any $x+h \in I_1$ has $\| h \| \leq \sqrt{n}(\frac{\delta}{r})$. Now $f(I_1)$ lies in a cube of edge $\frac{a}{r^{k+1}}$ with center at $f(x)$, where $a = 2C \cdot (\sqrt{n}\delta)^{k+1}$. Then $f(C_k \cap I^n)$ is contained in at most r^n cubes of total volume $\leq r^n(\frac{a}{r^{k+1}})^p = a^p r^{n-(k+1)p}$. Since $k+1 > \frac{n}{p}$, as $r \to \infty$ this volume tends to 0. Hence $f(C_k \cap I^n)$ has measure 0.

This completes the proof of Sard's theorem.

Remark 1) H. Whitney has given an example of a function of class C^{n-m}on an open subset $U \subset \mathbb{R}^n$ with $n > m$ such that the set of critical values is $\mathbb{R}^m$.

2) If $m \leq n$, then the set of regular values of a C^∞ map $U \subset \mathbb{R}^n \to \mathbb{R}^m$ is everywhere dense. Let $y \in \mathbb{R}^m$ be a regular value. Then either $f^{-1}(y) = \phi$

or it is a differentiable submanifold of dimension $n - m$.

§ 4.7. Applications of Sard's Theorem

1) Existence of Morse Functions Let M be a C^∞ manifold of dimension m and let $M \subset \mathbb{R}^n$ be a closed embedding for some $n > m$. For any point $p \in \mathbb{R}^n - M$, define the function $f : M \to \mathbb{R}$ by $f(q) = \| p - q \|^2$. Using Sard's Theorem we can show that for all points p outside a set of measure 0 the function f has only non-degenerate critical points, i.e., at any point $q \in M$, there are local coordinates $x_1, \cdots, x_m$ at q such that near q the matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j}(q))$ is non-singular, if all the partial derivates $\frac{\partial f}{\partial x_i}(q) = 0$.

The condition that $\frac{\partial f}{\partial x_i}(q) = 0$ for all i means q is a critical point of the distance function f defined on M. If $(\frac{\partial^2 f}{\partial x_i \partial x_j}(q))$ is non-singular then q is called a *non-degenerate critical point*. It is clear that f is a proper map $M \to \mathbb{R}$. The result just mentioned shows the existence of *Morse functions on* M.

2) Transversality Theorem Let P be a C^∞ -closed submanifold of a C^∞ manifold N. For any C^∞ manifold M a C^∞ map $g : M \to N$ is said to be transverse to P if whenever $p \in P \cap g(M)$ the tangent space $T_p(P)$ and $df(T_x(M))$ generate $T_p(N)$, where $p = f(x)$ for any $x \in f^{-1}(p)$.

Thom's transversality theorem says that for any C^∞ map $f : M \to N$, there is a C^∞-homotopic map $g : M \to N$ which is arbitrarily close to f such that g is transverse to P.

The proof of this result depends on Sard's theorem. An application of Rank Theorem show that if g is transverse to P then $g^{-1}(P)$ is a C^∞-submanifold of M of codimension same as the codimension of P in N.

In particular, if dim $M+$ dim $P <$ dim N, then $g^{-1}(P) = \phi$. Now let $f : S^m \to S^n$be a continuous map, where $m < n$. We can assume by approximation that f is a C^∞ map. Taking $P = \{p\}$, we can deform f to a C^∞ map $g : S^m \to S^n - \{p\}$. Since $S^n - \{p\}$ is homeomorphic to $\mathbb{R}^n$, we conclude that for $m < n, \pi_m(S^n) = (0)$.

3) Embedding in $\mathbb{R}^n$. Let M be a C^∞ manifold of dimension n. It is well-known that there is a closed embedding $M \subset \mathbb{R}^N$ for $N >> 0$. If $N > 2n+1$, we can find a direction $l \in \mathbb{R}^N$ such that the orthogonal projection to l maps M diffeomorphically into $\mathbb{R}^{N-1}$. Continuing this way, we can embed $M \subset \mathbb{R}^{2n+1}$. Further, we can show that any two embeddings of M in $\mathbb{R}^{2n+2}$ are isotopic, i.e., there is a C^∞ map $F : M \times I \to \mathbb{R}^{2n+2}$ such that for each $t \in I, F : M \times \{t\} \to \mathbb{R}^{2n+2}$ is an embedding and $F \mid M \times \{0\}, F \mid M \times \{1\}$ are given embeddings of M in $\mathbb{R}^{2n+2}$. This shows that any embedding $S^n \subset \mathbb{R}^{2n+2}$ is unknotted.

4) Tubular neighbourhoods. Let M be a compact C^∞ manifold with an embedding $M \subset \mathbb{R}^n$. Define for $\epsilon > 0$ $N_\epsilon(M) = \{y \in \mathbb{R}^n \mid d(y, M) \leq \epsilon\}$. Then for some $\epsilon > 0$, $N_\epsilon(M)$ is a smooth n-dimensional manifold in $\mathbb{R}^n$ with C^∞ boundary $\partial N_\epsilon(M)$, with a fiber bundle map $N_\epsilon(M) \to M$ whose fibers are discs D_{n-m} of dimension $n - m$, where $m = \dim M$. Further, $\partial N_\epsilon(M) \to M$ is a C^∞ sphere bundle with fiber S^{n-m-1}.

The proof of this is similar to the proof of existence of Morse function given earlier.

§ 5. Integration on manifolds

§ **5.1. Exterior differentiation** Let V be a C^∞ manifold. Let $A^p(V)$ denote the space of C^∞ differential p-forms on V. Hence $A^0(V)$ is the vector space of all C^∞ functions on V. Then there is a map $d : A^p(V) \to A^{p+1}(V)$ for each p such that

(1) d is $\mathbb{R}$-linear, i.e., $d(\alpha\omega_1 + \beta\omega_2) = \alpha d\omega_1 + \beta d\omega_2$ for any $\alpha, \beta \in \mathbb{R}$ and $\omega_1, \omega_2 \in \wedge^p(V)$.

(2) For $p = 0, d(f)$ is the 1-form such that

$$d(f)_a = \text{the image of } f \text{ in } T_a^*(V).$$

(3) $d(d\omega) = 0$ for $\omega \in A^p(V)$ for any p.

(4) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \omega_2$ for any $\omega_1 \in A^p(V), \omega_2 \in A^q(V)$.

The map d is described in a coordinate chart (U, φ) as follows. Let $x_1, \cdots, x_n$ be local coordinates in U. Then $\omega = \sum_{i_1 < \cdots < i_p} f_{i_1 \cdots i_p} dx_{i_1 \wedge \cdots \wedge dx_{i_p}}$. Define $d\omega = \sum_{i_1 < i_2 \cdots < i_p} d(f_{i_1 \cdots i_p}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Here $df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$. Only the properties (3) and (4) need to be verified. This is left to the reader.

Proposition 2. Let V, W be C^∞ manifolds and $f : V \to W$ a C^∞ map. Then for any $p \geq 0$ and $\omega \in A^p(W), d_{(V)} f^*(\omega) = f^*(d_{(W)}\omega)$.

Proof We assume that W is an open subset of $\mathbb{R}^m$. Now f^* is an algebra homomorphism of $\wedge T^*(W) \to \wedge T^*(V)$. Hence it suffices to prove the result when $p = 0$ or $p = 1$ and $\omega = dg, g \in C^\infty(W)$. If $p = 0$, then $\omega = g$ is a C^∞ function. By definition, $f^*((dg)_{f(a)}) = d(g \circ f)_a$.

If $\omega = dg$, where g is a function then

$$f^*(d(dg)) = 0 \text{ and } d[f^*(dg)] = d[d(g \circ f)] = 0.$$

§ 5.2. Orientation

Let V be a C^∞ manifold of dimension n. If $\exists$ a $C^\infty n-$ form ω on V which is nowhere 0 on V then ω is called an *orientation* on V and V is said to be orientable.

Proposition 3. V is orientable iff there exists a system of coordinates (U_i, φ_i) with $\cup U_i = V$ such that the transition functions $\varphi_i \circ \varphi_j^{-1} \mid_{\varphi_j(U_i \cap U_j)}$ have a positive jacobian determinant whenever $U_i \cap U_j \neq \phi$.

Proof Let ω be an orientation on V. For any $a \in V$ there is a coordinate chart (U_a, φ) at a such that $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ for $x \in U_a$ such that and $f(x) \neq 0$ in U_a. By interchanging x_1, x_2 we can assume that $f(x) > 0 \forall x \in V_a$.

If $y_1, \cdots, y_n$ are coordinates in an intersecting chart U_b, then in $U_a \cap U_b, \omega = \frac{f \cdot J(x_1, \cdots, x_n)}{(y_1, \cdots, y_n)} dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$. By assumption above, $fJ > 0$ $\forall p \in U_a \cap U_b$. It follows that $J > 0$ for all $p \in U_a \cap U_b$.

If $x_1^i, x_2^i, \ldots, x_n^i$ are local coordinates in the coordinate chart (U_i, φ_i) such that in $U_i \cap U_j$ the determinant $|\frac{J(x^i)}{(x^j)}| > 0$. Then we say that the system of local coorditates x^i in V are *positively oriented*.

Conversely, suppose (U_i, φ_i) are coordinate charts such that the Jacobian determinants $\frac{J(x_1, \cdots, x_n)}{(y_1, \cdots, y_n)} > 0$ in $U_i \cap U_j$. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$. Define $\omega_x = \sum_i \psi_i(x) dx_1^i \wedge \cdots \wedge dx_n^i$, where $x_1^i, \cdots, x_n^i$ are local coordinates on U_i.

Then ω_x is C^∞ and > 0 for any x and hence an orientation of V.

§ 5.3. The double cover

Consider the C^∞ bundle $\wedge^n T^*(V)$ and $E = \{\xi \in \wedge^n T^*(V) / \xi \neq 0\}$. Define $p(\xi) = a$ if $\xi \in \wedge^n T_a^*(V)$. Define an equivalence relation in E by $\xi_1 \sim \xi_2$ if $p(\xi_1) = p(\xi_2)$ and $\xi_1 = \lambda \cdot \xi_2$ for some $\lambda > 0$. Let $\tilde{V} = E / \sim$.

Proposition 4. $\tilde{V}$ is Hausdorff and $\tilde{V} \to V$ is a covering of degree 2.

Proof The equivalence relation is open and the graph of the equivalence relation in $E \times E$ is closed. Hence $\tilde{V}$ is Hausdorff.

Let (U, φ) be a coordinate chart and $a \in U$. Define ξ, η by $\xi_x = dx_1 \wedge \cdots \wedge dx_n, \eta_x = -dx_1 \wedge \cdots \wedge dx_n$.

Now $p^{-1}(U) = (\cup U_{x \in U} \overline{\xi}_x) U (U_{x \in U} \overline{\eta}_x)$, hence $\tilde{V}$ is a covering of degree 2.

Corollary 1 If V is connected then V is orientable if and only if $\tilde{V}$ is not connected.

Proof Suppose V is connected and orientable. Let ω be an orientation. Then $\cup U_{x \in V} \overline{\omega}_x$ is a non-empty open and closed subset of $\tilde{V}$ so that $\tilde{V}$ is not connected.

Suppose V is connected and $\tilde{V}$ is not connected. Let $\overline{\xi}_a \in \tilde{V}$ and U_a the connected component of $\overline{\xi}_a$ in $\tilde{V}$. Now $p : U_a \to V$ is a covering and it is a homeomorphism since $\tilde{V} \to V$ is a covering of degree 2. It follows that there is a section $s : V \to \tilde{V}$ whose image lies in one of the connected components of $\tilde{V}$ and this gives a nowhere vanishing $C^\infty n$- form on V. Thus V is orientable.

Corollary 2 If V is simply connected then it is orientable.

Remark $\tilde{V}$ is always orientable.

The pull back of the bundle $\wedge^n T^*(V) \to V$ to $\tilde{V}$ has a nowhere vanishing C^∞ section and the pull back bundle is nothing but $\wedge^n T^*(\tilde{V})$ (this is because the pull back of $\wedge^n T^*(V) \to V$ to E has a nowhere vanishing section).

§ 5.4. Manifolds with boundary

Let $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n / x_n \geq 0\}$.

Let V be a Hausdorff topological space such that V has an open cover $\{U_i\}$ with homeomorphism $\varphi_i : U_i \to \mathbb{R}^n_+$, onto an open subset of $\mathbb{R}^n_+$, such that whenever $U_i \cap U_j \neq \phi$, the map $\varphi_i \circ \varphi_j^{-1}/_{\varphi_j}(U_i \cap U_j)$ is a C^∞ map of $\varphi_j(U_i \cap U_j)$ as a subset of $\mathbb{R}^n_+$. Then V is called a C^∞ manifold with boundary.

For a real valued function on $\mathbb{R}^n_+$, we define $\frac{df}{\partial x_i}$ for $i < n$ in the same way as for a function on $\mathbb{R}^n$ and $\frac{df}{\partial x_n} \mid_a = \lim_{h \to +0} \frac{f(a_1, a_2, \cdots, a_n + h) - f(a_1, \cdots, a_n)}{h}$.

For a C^∞ manifold with boundary, C^∞ functions, tangent vectors, $T_a(V)$, differential forms etc. are defined in the same way as for a C^∞ manifold. Orientation is also defined in an analogous way.

We will assume that a C^∞ manifold with boundary is second countable.

In local coordinates $(U, \varphi), \varphi(U) \subset \mathbb{R}^n_+$ and $\varphi(a) = 0$, the set $\partial V = \{x \in V / x_n = 0\}$ is called the boundary of V. It is easy to see that ∂V is a C^∞ manifold of dimension $n - 1$.

For a point $a \in \partial V$, a cube Q associated to a coordinate chart (U, φ) has the property that

$\varphi(Q \cap \partial V) = \{(x) \in \mathbb{R}^n / -1 \leq x_i \leq 1 \; for \; i < n$ and $x_n = 0\}$. The *interior* Q^o is defined in this case as $\varphi^{-1}\{(x) \in \mathbb{R}^n / -1 < x_i < 1$ for $i \leq n-1$ and $0 \leq x_n < 1\})$. Hence in this case the interior of Q has a different meaning than for a cube contained in $V - \partial V$.

§ 5.5. Integration

Recall that a subset $A \subset \mathbb{R}^n$ has ($n-$dimensional) *Jordan content zero*, $C(A) = 0$, if for any $\epsilon > 0$ there is a finite collection of cubes $C_1, \cdots, C_\ell$ which cover A and whose total volume is less that ϵ. It is easy to see that if A is compact then $C(A) = 0$ iff A has Lebesgue measure zero.

Definition A bounded subset $D \subset \mathbb{R}^n$ is said to be a domain of integration if the boundary $\overline{B} - B^\circ$ has content zero.

A function f on $\mathbb{R}^n$ is *almost continuous* if the set of points at which f is not continuous has content zero.

Lemma 3. Given a domain D of integration in $\mathbb{R}^n$ and f a real valued, bounded, almost continuous function on D, the Riemann integral $\int_D f dx_1 dx_2 \cdots dx_n$ exists.

A function f as in the above lemma is said to be *integrable* on D.

Recall the following result on change of variables in integration.

Proposition 5. let $\varphi : U \to U'$ be a a diffeomorphism of a domain of integration $U \subset \mathbb{R}^n$ onto a domain of integration $U' \subset \mathbb{R}^n$ and let $J(\varphi)$ denote the Jacobian determinant of φ. Let f' be integrable on U'. Then $f := f' \circ \varphi$ is integrable on U and $\int_{U'} f' dy_1 \cdots dy_n = \int_U f'(\varphi(x)) \mid J(\varphi) \mid dx_1 \cdots dx_n$.

Using an argument similar to the proof of Sard's Theorem, we see easily the following.

Lemma 4. If $A \subset \mathbb{R}^n$ is a relatively compact subset of content zero and $\varphi : U \to \mathbb{R}^m$ a C^∞ map, where U is an open neighbourhood of A and $n \leq m$. The $\varphi(A)$ has content zero.

This result will enable us to extend the notion of content zero to C^∞ manifolds in an obvious manner.

Now let V be a C^∞ manifold, (U, φ) a coordinate chart. We assume that V is orientable and ω a nowhere vanishing n-form on V. We call ω *a volume form* on V. In $U, \omega = f dx_1 \wedge \cdots \wedge dx_n$, where f is a C^∞ function on U and $f(x) \neq 0$ for any $x \in U$. Let $\eta = g dx_1 \wedge \cdots \wedge dx_n$ be any C^∞ form on M whose support is compact and contained in U. Then $\eta = h \cdot \omega$, where h is a nowhere zero C^∞ function on V. Define

$$\int_V \eta = \int_U g dx_1 \wedge \cdots \wedge dx_n = \int_U h \cdot \omega.$$

If (U', φ') is another coordinate chart containing the support of η, then supp $\eta \subset U \cap U'$. Recall the change of variables formula for integration:

$$\int_U g(x) dx_1 \wedge dx_2 \cdots dx_n = \int_{U'} g \left| \frac{J(x)}{(y)} \right| dy_1 \wedge \cdots dy_n.$$

But $\frac{J(x_1,\cdots,x_n)}{(y_1,\cdots,y_n)} > 0$ since V is orientable. Hence $\int_V \eta$ is well-defined.

Now let η be an arbitrary C^∞ n-form on V. Let $\{U_i, \varphi_i)\}$ be a cover of V by coordinate charts such that the Jacobian determinants $\frac{J(x)}{(y)} > 0$ for any two overlapping U_i, U_j. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{U_i\}$. We can suppose that *supp* ψ_i is compact and contained in U_i and $\sum \psi_j = 1$. Now $\eta\psi_j$ are C^∞ forms defined on whole of V and support $\eta \cdot \psi_j \subset U_j$ is compact.

Define $\int_V \eta = \sum_j \int_{U_j} \eta\psi_j$.

This is independent of the choice of the partition of unity. For, let $\{h_j\}$ be another partition of unity subordinate to $\{U_j\}$. Now $\psi_j = \sum_\ell \psi_j h_\ell$ and support $\psi_j h_\ell \subset$ support $\psi_j \cap$ support h_ℓ.

$\int_V \psi_j \eta = \sum_\ell \int_V \psi_j h_\ell \eta$ and hence

$$\sum_j \psi_j \eta = \sum_\ell \int_V \sum_j \psi_j h_\ell \eta = \sum_\ell \int_V h_\ell \eta.$$

Lemma 5. Let $(V, \partial V)$ be a C^∞ manifold with boundary. If V is orientable then ∂V is orientable.

Proof We only need to check the following.

Let $p \in \partial V$ and $(U, \varphi), (U', \psi)$ coordinate charts near p. Let $x_1, \cdots, x_n$ be coordinates in U and $y_1, \cdots, y_n$ in U'. In U, ∂V is defined by $x_n = 0$ and in U' by $y_n = 0$. Thinking of $y_1, \cdots, y_n$ as functions of $x_1, \cdots, x_n$ in $U \cap U', y_n = x_n(a_0 + g(x_1, \cdots, x_n))$, where a_0 is a non-zero constant and $g(0, \cdots, 0) = 0$. It follows that $\frac{\partial y_n}{\partial x_j}(p) = 0$ for $j < n$. The determinant

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} (p) > 0$$

since V is orientable. The functions $x_1, x_2, \cdots, x_{n-1}$ restricted to $U \cap \partial V$ and

$y_1, y_2, \cdots, y_{n-1}$ restricted to $U' \cap \partial V$ define local coordinates on ∂V near p.

It follows that $\frac{\partial y_n}{\partial x_n}(p) \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_{n-1}}{\partial x_{n-1}} \\ \vdots & & \\ \frac{\partial y_{n-1}}{\partial x_1} & \cdots & \frac{\partial y_{n-1}}{\partial x_{n-1}} \end{vmatrix} (p) > 0.$

For points on $V - \partial V$ near $p, x_n > 0$ and $y_n > 0$. This implies that $a_0 > 0$ and hence $\frac{\partial y_n}{\partial x_n}(p) > 0$. Hence the determinant

$$\left| \frac{\partial y_i}{\partial x_j} \right|_{\substack{i<n \\ j<n}} \quad (p) > 0.$$

Thus $(U \cap \partial V, \varphi)$ give coordinate charts on ∂V such that the transition functions have Jacobian determinant > 0. Hence ∂V is orientable. Namely, if (x^i) are positively oriented local coordinates in U_i for each i such that $\{x^i_n = 0\}$ defines $\partial V \cap U_i$ then $x^i_1, x^i_2, \ldots, x^i_{n-1}$ give positively oriented local coordinated in the covering $\partial V \cap U_i$ of ∂. As in the proof of Proposition 3 in §5.2 this gives an orientation on ∂V.

§ 5.6. Stoke's Theorem

Let $(V, \partial V)$ be a compact oriented manifold with boundary of dimension n. By a cube Q in V we mean the following. If $Q \subset V - \partial V$, Q is contained in a coordinate chart (U, φ) and $Q = \{(x_1, \cdots, x_n) \mid -1 \leq x_i \leq 1 \ \forall i\}$. If $Q \cap \partial V \neq \phi$, then $\exists$ a coordinate chart (U, φ) containing Q such that $Q = \{(x_1, \cdots, x_n) \mid 0 \leq x_i \leq 1 \ for \ 1 \leq i \leq n-1, \ -1 \leq x_n \leq 1\}$ and $Q \cap \partial V = \{(x) \in Q \mid x_n = 0\}$.

The interior Q^o of Q is defined as follows. If $Q \subset V - \partial V$ then $Q^o = \{(x_1, \cdots, x_n) | -1 < x_i < 1 \ for \ all \ i\}$. If $Q \cap \partial V \neq \phi$ then $Q^o = \{(x_1, ..., x_n) | -1 < x_i < 1 \ for \ i < n, \ 0 \leq x_n < 1\}$.

We orient ∂V as follows. If n is even, let ∂V have the induced orientation and where n is odd let ∂V have the opposite orientation as that induced by the orientation of V.

Stoke's Theorem Let ω be an $(n-1)$-form of class C^∞ on V. Let ∂V be oriented as above. Then we have

$$\int_V d\omega = \int_{\partial V} \omega \mid_\partial .$$

In particular, if $\partial V = \phi$ then $\int_V d\omega = 0$.

Proof We can assume that ω has support contained in the interior Q^o of a cube $Q \subset (U, \varphi)$. Write

$$\omega = \sum_{j=1}^n (-1)^{j-1} \lambda_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n.$$

Now $d\omega = (\sum_{j=1}^n \frac{\partial \lambda_j}{\partial x_j}) dx_1 \wedge \cdots \wedge dx_n$. Thus

$$\int_V d\omega = \int_Q (\sum_{j=1}^n \frac{\partial \lambda_j}{\partial x_j}) dx_1 \wedge \cdots \wedge dx_n = \sum_j \int_{-1}^1 \cdots \int_{-1}^1 \frac{\partial \lambda_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n = \int_{-1}^1 \cdots \int_{-1}^1$$

$$[\lambda_j(x_1, \cdots, x_{j-1}, 1, x_{j+1}, \cdots, x_n) - \lambda_j(x_1, \cdots, x_{j-1}, -1, x_{j+1}, \cdots, x_n)] dx_1 \cdots \widehat{dx_j} \cdots dx_n.$$

Case 1 $Q \cap \partial V = \phi$.

Since supp $\omega \subset Q^o, \lambda_j(x) = 0$ if $x_j = -1$ or 1. Hence each integrand above vanishes and $\int_V d\omega = 0$. Since $\omega \mid_{\partial V} = 0$ also, $\int_{\partial V} \omega = 0$ and Stoke's theorem is proved in this case.

Case 2. $Q \cap \partial V \neq \phi$. Now $Q^o = \{(x) \mid -1 < x_i < 1$ for $i \leq n-1$ and $0 \leq x_n < 1\}$. In this case all the integrands in the summation are zero except the one corresponding to $j = n$. Hence

$$\int_V d\omega = -\int_{-1}^1 \cdots \int_{-1}^1 \lambda_n(x_1, \cdots, x_{n-1}, 0) dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Now $\omega \mid_{\partial V} = (-1)^{n-1} \int_{-1}^1 \cdots \int_{-1}^1 \lambda_n(x_1, \cdots, x_{n-1}, 0) dx_1 \wedge \cdots \wedge dx_{n-1}$. Using the convention about induced orientation on ∂M, we get the required result.

Special cases of Stoke's Theorem.

1) Green's theorem Let $V \subset \mathbb{R}^2$ be a compact manifold with boundary a union of simple closed curves.

If ω is a C^∞ 1-form on V, then using the natural coordinates x, y on $\mathbb{R}^2, \omega = adx + bdy$. Here a, b are restrictions of C^∞ functions on some open set containing V. Now $d\omega = (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx \wedge dy$. By Stoke's theorem, $\int_V (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx \wedge dy = \int_{\partial V} adx + bdy$.

For the integral on the right hand side, the orientation on ∂V is such that as we traverse the boundary, the region V is on the left. Hence the integral on the right is the line integral along the curve ∂V. This is the Green's Theorem.

2 Divergence theorem. Let $V \subset \mathbb{R}^3$ be a compact 3-manifold with boundary a union of C^∞ surfaces. Let $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ be a 2-form on V, where P, Q, R are C^∞ functions on some open set containing V. Now $d\omega = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz$. By Stoke's theorem,

$$\int_V (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz = \int_{-\partial V} Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy.$$

This is the divergence theorem.

3) Curl Let V be a C^∞ surface in $\mathbb{R}^3$ with boundary a union of simple closed curves. Any 1-form on V is of the form $\omega = Adx + Bdy + Cdz$, where A, B, C are C^∞ functions on V. Stoke's theorem asserts that

$$\int_V (\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z})dy \wedge dz + (\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})dx \wedge dy + (\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x})dz \wedge dx = \int_{\partial V} Adx + Bdy + Cdz.$$

The integrals on the left-hand side can be converted to surface integrals over V.

Remark The function $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ is called the *divergence* of the vector field (P, Q, R) on V°. Similarly the vector $(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}, \frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}, \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y})$ is called the *curl* of the vector field (A, B, C).

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