Differential Geometry and Topology

Advanced Foundation School-1,Pune

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1. Let I be the closed unit interval $[0,1]$ of real numbers equipped with the standard topology. Let $H(I)$ denote the group of all homeomorphisms of I. Show that $H(I) = H_0(I) \cup H_1(I)$ where

   $H_0(I) = \Big\{ f : I \to I | f$ is a strictly increasing continuous function such that $f(0) = 0$, and $f(1) = 1. \Big\}$

   $H_1(I) = \Big\{ f : I \to I | f$ is a strictly decreasing continuous function such that $f(0) = 1, and f(1) = 0. \Big\}$

2. Let $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$

   (a) Show that $\mathbb{D}^2$ with an oriented 1-handle attached is an annulus.

   (b) What is $\mathbb{D}^2$ with a non-orientable 1-handle attached?

3. (Handle-Sliding Process): Let Z be the topological space obtained from D^2 by attaching two disjoint non-oriented 1-handles. Let W be the topological space obtained from D^2 by attaching two intertwined 1-handles, one orientable, and the other non-orientable. Show that Z is homeomprphic to W

4. Let S_1 and S_2 be two surfaces ,define $S_1 \# S_2$.What is $\mathbb{P}^2(\mathbb{R}) \# \mathbb{P}^2(\mathbb{R})$?

5. Show that $\mathbb{P}^2 \# K^2 \cong \mathbb{P}^2 \# T^2$,where T^2 is the torus and K^2 is the Klein bottle.

6. Show that X_g = $4g$-gon with a certain identification of edges on the boundary.

7. Show that Y_h = $2h$-gon with a certain identification of edges on the boundary.

8. Consider the map $\sigma : \mathbb{R}^3 - 0 \to \mathbb{R}^3 - 0$ given by $\sigma(\bar{x}) = -(\bar{x})$. Consider X_g's embedded in $\mathbb{R}^3$ invariant under σ. Show that $X_g/\sigma = Y_h$ where $2 - 2g = 2(1 - h)$.

   (This generalizes the standard construction which shows S^2 as a double cover of $\mathbb{P}^2(\mathbb{R})$.)

9. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a smooth function and define $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ by $\gamma(t) = (t, g(t))$.Compute the curvature of γ.

10. Show that a directed curve in $\mathbb{R}^3$ is a straight line if and only if all its tangent lines are parallel.

11. (a) Compute the curvature of $\gamma(t) = (t, t^3)$
    (b) Show that for a regular curve γ with curvature $\kappa(t) \gneq 0, \forall t$,the curvature κ is a smooth function of t
    (c) Is the above statement true if $\kappa(t) = 0$ for some $t \in \mathbb{R}$?

12. For the following curves ,analyse the behaviour of the curvature (i.e. try to <u>see</u> where the curvature attains maxima and minima and also find on which side of the tangent line the curve lies as you move in the increasing direction of t)

    (a) $\gamma(t) = (acost, bsint), a \gneq b \gneq 0$
    (b) $\sigma(t) = (t, t^2)$
    (c) $\nu(t) = (cosht, sinht)$

    (Verify your guesses by computing the curvatures explicitly)

13. Let $X = \mathbb{R}$ as a set.Consider the function $\phi\colon X \to \mathbb{R}$ given by $\phi(x) = x^3$

    (a) Prove that ϕ defines a smooth structure on X.
    (b) With the smooth structure as defined above is the identity map $i\colon X \to \mathbb{R}$ a diffeomorphism?

14. Let $\gamma\colon (0,1) \to \mathbb{R}^3$ be a curve in the xz-plane i.e. $\gamma(t) = (x(t), 0, z(t))$.Assume $x(t) \gneq 0$. We revolve γ around the z-axis to get a <u>Surface of Revolution</u>

    (a) Write down a parametrization for a surface of revolution.
    (b) Describe the surface of revolution of the curve $\gamma(t) = (cosht, 0, t)$

15. (a) Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^1 function s.t. $0 \in f(\mathbb{R}^{n+1})$ and $S := f^{-1}(0)$,assume $\nabla f(p) \neq 0 \forall p \in S$
    Use the Inverse Function Theorem to show that S is an n-manifold.

(b) Describe $T_p(S), p \in S$.

16. Let $S_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A = A^t\}$; $S_n(\mathbb{R})$ is a vector space of dimension $n(n+1)/2$.

    (a) Show that $F\colon M_n(\mathbb{R}) \to S_n(\mathbb{R})$ given by $F(X) = XX^t$, is smooth.

    (b) Let $O(n) = F^{-1}(I)$,where $I \in S_n(\mathbb{R})$ is the identity matrix.Show that $O(n)$ is a manifold of dimension $n(n-1)/2$.

17. $f\colon \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is given by ;$f(x_1, \cdots, x_n) = \frac{1}{r}(x_1, \cdots, x_n)$
    Compute the jacobian and interpret geometrically.

18. Let M an n-dimensional manifold and $p \in M$.

    (a) Show that for any $v \in T_p(M), \exists$ a smooth curve γ with $\gamma(0) = p$ and $\gamma'(0) = v$.

    (b) If (U, x) is a chart around p,then we can write any $v \in T_p(M)$ as $v = \sum_{i=1}^n v(x_i) \frac{\partial}{\partial x_i} |_p$ .If (V, y) is another chart around p ,how are the expressions for $v \in T_p(M)$ w.r.t (U, x) and (V, y) related?

19. Let ω be an n-form on the n-manifold S ;
    (a) Show that if $\{v_1, \cdots, v_k\}$ is a linearly dependent set in $T_p(S), p \in S$ then $\omega(v_1, \cdots, v_k) = 0$.
    (b) Show that if $k \gneq n$ then ω is identically zero.

20. $S \subset \mathbb{R}^{n+1}$ is an oriented n-manifold.Let ω_1 be a k-form and ω_2 an l-form on S. Their exterior product is defined by ,

    $$\omega_1 \wedge \omega_2(v_1, \cdots, v_{k+l}) := \frac{1}{k!l!} \sum (sign\sigma) \omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \omega_2(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)})$$

    where the sum is taken over all permutations σ of $\{1, \cdots, k+l\}$

    (a) Show that $\omega_1 \wedge \omega_2$ is a (k+l)-form on S.

    (b) Show that $\omega_2 \wedge \omega_1 = (-1)^{kl} \omega_1 \wedge \omega_2$

    (c) Show that if ω_3 is another l-form on S ,then $\omega_1 \wedge (\omega_2 + \omega_3) = (\omega_1 \wedge \omega_2) + (\omega_1 \wedge \omega_3)$

    (d) Show that if ω_3 is an m-form on S; $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3)$

21. M and N are smooth manifolds and $f\colon M \to N$ is a smooth map.Let ω be a k-form on N. The pull back of ω is defined by;

    $(f^*\omega)(v_1, \cdots, v_k) = \omega(f_*(v_1), \cdots, f_*(v_k))$ where f_* is the differential of f;

(a) Show that $f^*\omega$ is a k-form on M

(b) $f\colon \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (x, -x)$ and $\omega = dx + dy$,compute $f^*\omega$.

22. Let M be a manifold of dimension m .

    (a) Let ω be a 1-form on M,find local expressions for ω

    (b) Express any k-form ϕ in terms of local coordinates.

23. (a) Define the Exterior Derivative of a k-form on M.

    (b) If f is a 0-form i.e. a smooth function on M,what is the exterior derivative df of f?

    (c) Show that for a k-form ω and an l-form η;
    $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

24. (a) Show that for $S^n(r)$,the n-dimensional sphere of radius $r \gneq 0$ in $\mathbb{R}^{n+1}$,the normal at $x \in S^n(r)$ is given by $N_x = \frac{x}{r}$

    (b) Let $\tilde{D}$ be the affine connection on $\mathbb{R}^n$ given by $\tilde{D}_v w = \sum v(w_i)\frac{\partial}{\partial x_i}$ where $w = \sum w_i \frac{\partial}{\partial x_i}$;

    i. For any n-manifold $S \subset \mathbb{R}^n$;can we define a connection by setting $D_{v_p} w := \tilde{D}_{v_p} w$?Analyse the case of $S^1 \subset \mathbb{R}^2$

    ii. How do we modify the above definition to get an affine connection on any hypersurface?

25. If ∇ is an affine connection on $\mathbb{R}^n$ s.t. for vector fileds u, v and w
    $w < v, w > = < \nabla_u v, w > + < v, \nabla_u w >$;
    Then show that ∇ is the same as the standard connection on $\mathbb{R}^n$.

26. Let (M, g) be a Riemannian manifold and $i\colon S \to M$ a submanifold.

    (a) Define the metric on S induced by the metric g.Show that the induced metric is smooth.

    (b) If $\mathbb{R}^{n+1}$ has the standard metric ,what is the induced metric on S^n?

    (c) What is the induced metric on a surface of revolution in $\mathbb{R}^3$?

27. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ and $\phi\colon \mathbb{R}^2 \to S$ be defined by $\phi(x, y) = (cosx, sinx, y)$ .Check that ϕ is a local isometry.Is it an isometry?