

# Linear Algebra : A Dynamical Viewpoint

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Let  $k$  be a field,  $V$  a vector space over  $k$  and  $T: V \longrightarrow V$  a linear transformation. *We do not assume that  $\dim_k V \lesssim \infty$ .*

We regard  $T$  as a dynamical system. This means that since the domain and the range of  $T$  are the same, we can iterate and obtain a sequence of operators;

$$T^0 = I, T^1 = T, T^2 = T \circ T, T^3 = T \circ T \circ T, \dots$$

Moreover, we have a canonical  $k$ -algebra homomorphism,

$$\varphi: k[X] \longrightarrow k[T]$$

- a)  $\varphi$  is clearly surjective.  $\ker \varphi = (m(X))$  where  $m(X)$  is either 0, or it may be chosen uniquely as a monic polynomial. In case  $\dim_k V \lesssim \infty$ ,  $m(X) \neq 0$ . In the following, we may assume  $m(X) \neq 0$  ( $\dim_k V$  may be  $\infty$ ).
- b) By definition if  $f(X) \in k[X]$  is such that  $f(T) = 0$ , then  $m(X)$  divides  $f(X)$ .  $m(X)$  is called the minimal polynomial of  $T$ . From a dynamical viewpoint, it is more basic than the characteristic polynomial, which is defined when  $\dim_k V \lesssim \infty$ .
- c) Basic Lemma :Let  $f(X) \in k(X)$  be such that  $f(T) = 0$ ,  
Suppose  $f(X) = r(X)s(X)$ ,  $\gcd(r(X), s(X)) = 1$ . Let  $R = r(T)$  and  $S = s(T)$ , then
  - (i)  $\text{im } R = \ker S$ ,  $\text{im } S = \ker R$ .
  - (ii)  $V = \ker R \oplus \ker S$  is a  $T$ -invariant decomposition.
  - (iii) Let  $T_r = T|_{\ker R}$ ,  $T_s = T|_{\ker S}$  then, the minimal polynomial of  $T_r$  (resp  $T_s$ ) divides  $r(X)$  (resp  $s(X)$ ). In particular if  $f(X) = m(X)$  then  $r(X)$  (resp  $s(X)$ ) is the minimal polynomial of  $T_r$  (resp  $T_s$ ).

Proof : Everything follows from  $\exists a(X), b(X) \in k[X]$  s.t.

$$a(X)r(X) + b(X)s(X) = 1$$

i.e  $\forall v \in V$ ,

$$a(T)r(T)v + b(T)s(T)v = v$$

So  $V = \text{im } R + \text{im } S$ ,

$\ker R \subseteq \text{im } S$ ,  $\ker S \subseteq \text{im } R$  and  $\ker S \cap \ker R = \phi$ .

Since  $f(T) = 0$  we also have  $\text{im } R \subseteq \ker S$ ,  $\text{im } S \subseteq \ker R$ .

q.e.d

Corollary: To understand the dynamics of  $T$  we reduce to the case :  
 $m(X) = p(X)^d$ , where  $p(X)$  is irreducible.

d) Special Case :  $m(X) = p(X)$  is irreducible.

Then  $k[X]/(p(X)) = K$  is a field and  $V$  has a  $K$ -structure, i.e.  $V$  can be regarded as a vector space over  $K$ .

Moreover let  $\alpha = [X]$  = the class of  $X$  in  $K$ . Then  $\alpha$  is a primitive element of  $K/k$ , i.e  $K = k[\alpha]$ .

Let  $\mu_\alpha : v \mapsto \alpha v$  be the scalar multiplication by  $\alpha$ . The dynamics of  $T$  on  $V$  is simply  $\mu_\alpha$ . So  $T$  looked through the “ $K$ -window” is very simple.

Remark : Let  $k = \mathbb{Q}$  and  $\dim_k V \leq \infty$ . Then in a probabilistic sense which can be made precise, “almost always” the minimal polynomial equals the characteristic polynomial, and it is irreducible.

Concretely : Let  $T = (a_{ij}) \in M_n(\mathbb{Q})$ .

$\chi_T(X) = p(X)$ , the characteristic polynomial of  $T$ .

Then in a generic sense we understand the dynamics of  $T$  via field theory. Namely, if  $K = k[X]/(p(X))$ , and  $\alpha$  = a root of  $p(X)$  in  $K$ . Then the dynamic model for  $(\mathbb{Q}^n, T)$  is  $(K, \mu_\alpha)$ .

Remark : In text books, most matrices in the examples have rational entries. The above remark shows that the exercises dealing with minimal polynomials are rather artificial, i.e. they deal with non-generic cases. Also if we merely consider them as special cases of matrices with entries in  $\mathbb{C}$  we lose a valuable arithmetic aspect.

- e) General Case :  $m(X) = p(X)^d$ , with  $p(X)$  irreducible,  $d \geq 1$ .  
 Let  $V_i = \ker p(T)^i$ ,  $i = 1, 2, \dots, d$  and set  $V_0 = 0$ .  
 We have a  $T$ -invariant filtration

$$0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_d = V$$

Let,  $\overline{T}_i$  be the operator induced by  $T$  on  $V_i/V_{i-1}, i=1, 2, \dots, d$   
 Then, the minimal polynomial of  $\overline{T}_i$  on  $V_i$  is  $p(X)$  and the discussion  
 in the special case (d) is valid for  $(V_i, \overline{T}_i)$ .

Let  $K = k[X]/(p(X))$  and  $\alpha =$  a root of  $p(X)$  in  $K$ . If  $k$  is perfect,  
 the  $K$ -structures on  $V_i$ 's can be lifted to a  $T$ -invariant  $K$ -structure on  
 $V$ , so that the minimal polynomial of  $T$  on  $V$  is  $(X - \alpha)^d$ .

This point is related to the ‘‘Jordan-Chevalley normal form’’. There is  
 an error on Bourbaki, Algebra volume 4 on this point.