

Regional Mathematical Olympiad - 2005
Maharashtra and Goa Region

Time: 3 hours

December 04, 2005

1. For each positive integer n , let $T_n = n(n+1)/2$. Find all pairs (n, m) of positive integers such that $n > m$ and $T_n - T_m = 2^k$ for some positive integer k . (8)
2. Find all values of a for which the equation $x^3 + x^2 - x + a = 0$ has three integer roots. (10)
3. A permutation $P = a_1 a_2 \cdots a_n$ of the set $\{1, 2, \dots, n\}$ is called special if there is exactly one j such that $1 \leq j \leq n - 1$ and $a_j > a_{j+1}$. For example, with $n = 5$, the permutations 13245, 51234, 24135 are special. Prove that the number of special permutations of the set $\{1, 2, \dots, n\}$ is $2^n - 1 - n$. (10)
4. Find all positive integers a, b such that the numbers $\frac{a^2 + b}{b^2 - a}$ and $\frac{b^2 + a}{a^2 - b}$ are both integers. (15)
5. In convex quadrilateral $ABCD$, $\angle ABC$ is obtuse and $\angle CAB = \angle DBC$. Also, the sides BC, AD and diagonal AC are of lengths which satisfy $BC^2 + AD^2 = AC^2$. Prove that $\angle ADB = \angle DCA$. (12)
6. For each number in the set $\{n + 1, n + 2, \dots, 2n\}$, consider its largest odd divisor and add all such largest odd divisors. Prove that the sum so obtained is n^2 . (15)
7. If x, y, z are positive numbers such that $x + y + z = 1$, prove that $\frac{1+x}{1-x} \cdot \frac{1+y}{1-y} \cdot \frac{1+z}{1-z} \geq 8$. (15)
8. On sides AC and BC of an acute-angled triangle ABC , rectangles $ACPQ$ and $BKLC$ are constructed outwardly. Assuming that these rectangles have equal areas, prove that the vertex C , the circumcentre O of $\triangle ABC$, and the midpoint M of segment PL are collinear. (15)