

**Regional Mathematical Olympiad - 2005**  
**Maharashtra and Goa Region**

Time: 3 hours

December 04, 2005

1. For each positive integer  $n$ , let  $T_n = n(n+1)/2$ . Find all pairs  $(n, m)$  of positive integers such that  $n > m$  and  $T_n - T_m = 2^k$  for some positive integer  $k$ . (8)
2. Find all values of  $a$  for which the equation  $x^3 + x^2 - x + a = 0$  has three integer roots. (10)
3. A permutation  $P = a_1 a_2 \cdots a_n$  of the set  $\{1, 2, \dots, n\}$  is called special if there is exactly one  $j$  such that  $1 \leq j \leq n - 1$  and  $a_j > a_{j+1}$ . For example, with  $n = 5$ , the permutations 13245, 51234, 24135 are special. Prove that the number of special permutations of the set  $\{1, 2, \dots, n\}$  is  $2^n - 1 - n$ . (10)
4. Find all positive integers  $a, b$  such that the numbers  $\frac{a^2 + b}{b^2 - a}$  and  $\frac{b^2 + a}{a^2 - b}$  are both integers. (15)
5. In convex quadrilateral  $ABCD$ ,  $\angle ABC$  is obtuse and  $\angle CAB = \angle DBC$ . Also, the sides  $BC, AD$  and diagonal  $AC$  are of lengths which satisfy  $BC^2 + AD^2 = AC^2$ . Prove that  $\angle ADB = \angle DCA$ . (12)
6. For each number in the set  $\{n + 1, n + 2, \dots, 2n\}$ , consider its largest odd divisor and add all such largest odd divisors. Prove that the sum so obtained is  $n^2$ . (15)
7. If  $x, y, z$  are positive numbers such that  $x + y + z = 1$ , prove that  $\frac{1+x}{1-x} \cdot \frac{1+y}{1-y} \cdot \frac{1+z}{1-z} \geq 8$ . (15)
8. On sides  $AC$  and  $BC$  of an acute-angled triangle  $ABC$ , rectangles  $ACPQ$  and  $BKLC$  are constructed outwardly. Assuming that these rectangles have equal areas, prove that the vertex  $C$ , the circumcentre  $O$  of  $\triangle ABC$ , and the midpoint  $M$  of segment  $PL$  are collinear. (15)