Problem Set 1: Rings and Modules

Throughout the following exercises A denotes a ring.

1. Let $\phi : A \to B$ be a homomorphism of rings. If J is an ideal of B, then show that $\phi^{-1}(J)$ is an ideal of A. Further, show that $J \in \operatorname{Spec}(B)$ implies $\phi^{-1}(J) \in \operatorname{Spec}(A)$. Is it true that $J \in \operatorname{Max}(B)$ implies $\phi^{-1}(J) \in \operatorname{Max}(A)$? Also, is it true that if I is an ideal of A, then $\phi(I)$ is an ideal of B? What if ϕ is surjective? Further, if ϕ is surjective, then is it true that $I \in \operatorname{Spec}(A)$ implies $\phi(I) \in \operatorname{Spec}(B)$, and that $I \in \operatorname{Max}(A)$ implies $\phi(I) \in \operatorname{Max}(B)$? Justify your answers.

2. Let I be an ideal of A and $q : A \to A/I$ be the natural homomorphism given by $x \mapsto x + I$. Show that $J \mapsto q(J)$ defines a bijective map from the ideals of A containing I and the ideals of A/I. Further, show that this bijection preserves inclusions, primality and maximality.

3. Assume that A is a PID. Given any $a, b \in A$, let $d = \mathrm{GCD}(a, b)$ and $\ell = \mathrm{LCM}(a, b)$. If $I = (a)$ and $J = (b)$, then show that
$$IJ = (ab), \quad I \cap J = (\ell), \quad I + J = (d), \quad \text{and} \quad (I : J) = (a/d).$$
Are these results valid if A is an arbitrary ring. What if A is a UFD? Justify your answer.

4. Consider ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ of A and the following three equalities.
$$\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}, \quad (\mathfrak{a} + \mathfrak{b})(\mathfrak{a} \cap \mathfrak{b}) = \mathfrak{a}\mathfrak{b}, \quad \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}) + (\mathfrak{a} \cap \mathfrak{c}).$$
In each case, determine if the equality is valid for arbitrary $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. If yes, then give a proof; otherwise give a counterexample. Also, if the answer is no, then determine if either of the inclusions $\subseteq$ and $\supseteq$ is valid, in general.

5. If I is an ideal of A, then show that $\sqrt{I}$ is an ideal of A.

6. Show that colons commute with intersections, whereas radicals commute with finite intersections. More precisely, if $\{I_\alpha : \alpha \in \Lambda\}$ is a family of ideals of a ring A and J is any ideal of A, then show that
$$\bigcap_{i \in \Lambda} (I_i : J) = \Big(\bigcap_{i \in \Lambda} I_i : J \Big) \quad \text{and if } \Lambda \text{ is finite, then} \quad \sqrt{\bigcap_{i \in \Lambda} I_i} = \bigcap_{i \in \Lambda} \sqrt{I_i}.$$
Give examples to show that these results do not hold (for finite families) if intersections are replaced by products.

7. Suppose A is not the zero ring and let $\mathfrak{N}$ be the nilradical of A. Show that the following are equivalent.

   (i) A has exactly one prime ideal.

   (ii) Every element of A is either a unit or a nilpotent.

   (iii) $A/\mathfrak{N}$ is a field.

8. Let k be a field. Show that $\dim_k k[X_1, \ldots, X_n]_d = \binom{n+d-1}{d}$.

9. Let $f(X) = a_0 + a_1X + \ldots + a_nX^n \in A[X]$. Prove the following:

   (i) f is a unit in $A[X]$ if and only if a_0 is unit in A and $a_1, a_2, \ldots, a_n$ are nilpotent in A.

   (ii) f is a nilpotent in $A[X]$ if and only if $a_0, a_1, \ldots, a_n$ are nilpotent in A.

   (iii) f is a zero divisor in $A[X]$ if and only if there exists $a \in A$ with $a \neq 0$ such that $af = 0$.

10. Let S be any multiplicatively closed subset of A. Consider the relation on $A \times S$ defined by $(a, s) \sim (b, t) \iff (at - bs) = 0$. Determine if $\sim$ is an equivalence relation.

11. Given any $f \in A$, let $S = \{f^n : n \in \mathbb{N}\}$ and $A_f = S^{-1}A$. Show that A_f is isomorphic to $A[X]/(Xf - 1)$.

12. Let S and T be multiplicatively closed subsets of A with $S \subseteq T$ and let U denote the image of T under the natural map $\phi : A \to S^{-1}A$. Show that $T^{-1}A$ is isomorphic to $U^{-1}(S^{-1}A)$.

13. Show that localization commutes with taking homomorphic images. More precisely, if I is an ideal of a ring A and S is a multiplicatively closed subset of A, then show that $S^{-1}A/S^{-1}I$ is isomorphic to $\overline{S}^{-1}(A/I)$, where $\overline{S}$ denotes the image of S in A/I.

14. Let A be an integral domain. Fix a quotient field K of A and consider the localization $A_{\mathfrak{p}}$, where $\mathfrak{p} \in \operatorname{Spec}(A)$, as subrings of K. Show that

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \in \operatorname{Max}(A)} A_{\mathfrak{m}}.$$

15. Consider the following ring-theoretic properties that A can have: (i) integral domain, (ii) field, (iii) PIR, (iv) PID, and (v) UFD. For each of these, determine if the property is preserved under the passage from A to a (i) residue class ring, (ii) polynomial ring, or (iii) localization.

16. Let M be an A-module and S be a multiplicatively closed subset of A. Define carefully the localization $S^{-1}M$ of M at S. With ideals replaced by A-submodules, determine which of the notions and results concerning localization of rings have an analogoue in the setting of modules.

17. Let $(A, \mathfrak{m})$ be a local ring [which means that A is a local ring and $\mathfrak{m}$ is its unique maximal ideal] and M be a finitely generated A-module. For $x \in M$, let $\overline{x}$ denotes the image of x in the $A/\mathfrak{m}$-module $M/\mathfrak{m}M$. Given any $x_1, \ldots, x_r \in M$, show that $\{x_1, \ldots, x_r\}$ is a minimal set of generators of M if and only if $\{\overline{x}_1, \ldots, \overline{x}_r\}$ is a basis for the $A/\mathfrak{m}$-vector space $M/\mathfrak{m}M$. Deduce that any two minimal set of generators of M have the same cardinality, namely, $\dim_{A/\mathfrak{m}} M/\mathfrak{m}M$.

18. Assume that A is not the zero ring and let $m, n \in \mathbb{N}$. Use Exercise 17 to show that A^m and A^n are isomorphic as A-modules iff $m = n$.

19. Given any $f, g \in A$, show that the principal open sets D_f and D_g of $\operatorname{Spec} A$ satisfy the following.
(i) $D_f = \emptyset \Longleftrightarrow f$ is nilpotent, (ii) $D_f = \operatorname{Spec}(A) \Longleftrightarrow f$ is a unit,
(iii) $D_f \cap D_g = D_{fg}$, and (iv) $D_f = D_g \Longleftrightarrow \sqrt{(f)} = \sqrt{(g)}$.

20. Given any $f \in A$, show that the principal open set D_f is quasi-compact. Further show that an open subset of $\operatorname{Spec}(A)$ is quasi-compact if and only if it is a finite union of principal open sets.

Problem Set 2: Noetherian Rings and Modules

Throughout the following exercises A denotes a ring.

1. Let $A = k[X_1, X_2, \ldots]$ be the polynomial ring in infinitely many variables with coefficients in a field k. Prove that A is not noetherian.

2. Let $\mathfrak{q}$ be an ideal of A and $\mathfrak{p} = \sqrt{\mathfrak{q}}$. Show that if A is noetherian, then $\mathfrak{p}^n \subseteq \mathfrak{q}$ for some $n \in \mathbb{N}$. Is this result valid if A is not noetherian? Justify your answer.

3. Let $\mathfrak{q}$ be a nonunit ideal of A. Show that $\mathfrak{q}$ is primary if and only if every zerodivisor in $A/\mathfrak{q}$ is nilpotent.

4. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and x be an element of A. Show that if $x \in \mathfrak{q}$, then $(\mathfrak{q} : x) = (1)$, whereas if $x \notin \mathfrak{q}$, then $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary, and in particular, $\sqrt{(\mathfrak{q} : x)} = \mathfrak{p}$. Further show that if $x \notin \mathfrak{p}$, then $(\mathfrak{q} : x) = \mathfrak{q}$.

5. Show that if $\mathfrak{q}$ is an ideal of A such that $\sqrt{\mathfrak{q}} \in \operatorname{Max}(A)$, then $\mathfrak{q}$ is primary.

6. Let $A = \mathbb{Z}[X]$ and consider the ideals $\mathfrak{q} = (4, X)$ and $\mathfrak{m} = (2, X)$. Show that $\mathfrak{m}$ is a maximal ideal of A and $\mathfrak{q}$ is $\mathfrak{m}$-primary, but $\mathfrak{q}$ is not a power of $\mathfrak{m}$.

7. Let $A = k[X, Y]$ and $I = (X^2, XY, Y^2)$. Show that $I = (X^2, Y) \cap (X, Y^2)$ is a primary decomposition of I. Is this an irredundant primary decomposition of I? Justify your answer.

8. Let I be a radical ideal and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_h$ be an irredundant primary decomposition of I, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$–primary for $1 \le i \le h$. Show that $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_h$. Deduce that I has no embedded component, and that $\mathfrak{q}_i = \mathfrak{p}_i$ for $1 \le i \le h$.

9. Given an ideal I of A, define $\mathcal{Z}(A/I) := \{x \in A : (I : x) \neq I\} \cup \{0\}$. Show that $\mathcal{Z}(A/I)$ is the union of the associated primes of I, that is,

$$\mathcal{Z}(A/I) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(A/I)} \mathfrak{p} \quad \text{and deduce that} \quad \mathcal{Z}(A) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(A/(0))} \mathfrak{p},$$

where $\mathcal{Z}(A)$ denotes the set of all zerodivisors of A.

10. Let I be a nonunit ideal of A and $\operatorname{Ass}(A/I)$ be the set of associated primes of I in A. Show that the minimal elements in $\operatorname{Ass}(A/I)$ are precisely the minimal elements in the set $V(I) = \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \supseteq I\}$ of primes containing I.

11. Let S be a multiplicative closed subset of A and $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of A. Show that if $S \cap \mathfrak{p} \neq \emptyset$, then $S^{-1}\mathfrak{q} = S^{-1}A$, whereas if $S \cap \mathfrak{p} = \emptyset$, then $S^{-1}\mathfrak{q}$ is $S^{-1}\mathfrak{p}$-primary and $S^{-1}\mathfrak{q} \cap A = \mathfrak{q}$. Deduce that if I is any ideal of A and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_h$ is a primary decomposition of I in A, then

$$S^{-1}I = \bigcap_{\mathfrak{p}_i \cap S = \emptyset} S^{-1}\mathfrak{q}_i \quad \text{and} \quad S^{-1}I \cap A = \bigcap_{\mathfrak{p}_i \cap S = \emptyset} \mathfrak{q}_i \, .$$

12. Let $A = k[X, Y, Z]/(XY - Z^2)$ and write x, y, z for the images of X, Y, Z in A, respectively. Show that $\mathfrak{p} = (x, z)$ is a prime ideal of A, but $\mathfrak{p}^2 = (x^2, xz, z^2)$ is not primary. Further show that $x \notin \mathfrak{p}^2$, but $x \in \mathfrak{p}^{(2)}$.

13. Given an ideal I of A, let $I[X]$ denote the set of all polynomials in $A[X]$ with coefficients in I. Show that $IA[X] = I[X]$. Further show that

    (i) $\mathfrak{p} \in \mathrm{Spec}\,(A) \Longrightarrow \mathfrak{p}[X] \in \mathrm{Spec}\ (A[X])$.

    (ii) $\mathfrak{q}$ is $\mathfrak{p}$-primary $\Longrightarrow \mathfrak{q}[X]$ is $\mathfrak{p}[X]$-primary.

    (iii) $\mathfrak{p}$ is a minimal prime of $I \Longrightarrow \mathfrak{p}[X]$ is a minimal prime of $I[X]$.

    (iv) $I = \cap_{i=1}^{n} \mathfrak{q}_i$ a primary decomposition of I
    $\Longrightarrow I[X] = \cap_{i=1}^{n} \mathfrak{q}_i[X]$ a primary decomposition of $I[X]$.

14. Let Δ be a simplicial complex with vertex set $V = \{1, 2, \ldots, n\}$, and let $F_1, F_2, \ldots, F_m$ be the facets (i.e., maximal faces) of Δ. Let I_Δ be the ideal of $k[X_1, \ldots, X_n]$ generated by the squarefree monomials $X_{i_1} \ldots X_{i_r}$ for which $\{i_1, \ldots, i_r\} \notin \Delta$. Given any face F of Δ, let P_F be the ideal of $k[X_1, \ldots, X_n]$ generated by the variables $X_{j_1}, \ldots, X_{j_s}$, where $\{j_1, \ldots, j_s\} = V \setminus F$. Prove that each P_F is a prime ideal and $I_\Delta = P_{F_1} \cap \cdots \cap P_{F_m}$ is an irredundant primary decomposition of I_Δ.

15. Let J be a monomial ideal of $k[X_1, \ldots, X_n]$ and u, v be relatively prime monomials in $k[X_1, \ldots, X_n]$. Show that $(J, uv) = (J, u) \cap (J, v)$. Also show that if $e_1, \ldots, e_n$ are positive integers, then $(X_1^{e_1}, \ldots, X_n^{e_n})$ is $(X_1, \ldots, X_n)$–primary. Use these facts to determine the associated primes and a primary decomposition of the ideal (X^2YZ, Y^2Z, YZ^3) of $k[X, Y, Z]$.

16. Consider $\mathbb{Q}/\mathbb{Z}$ as a $\mathbb{Z}$-module. Determine if it is a noetherian module?

Problem Set 3: Dimension, Height, and Integral Extensions

Throughout the following exercises A denotes a ring.

1. Assume that A is a noetherian ring. If $a \in A$ is a nonzerodivisor and $\mathfrak{p}$ is a minimal prime of (a), then prove that $\operatorname{ht} \mathfrak{p} = 1$.

2. Give an example of a minimal prime $\mathfrak{p}$ of a principal ideal of a noetherian ring such that $\operatorname{ht} \mathfrak{p} = 0$.

3. Prove the converse of Krull's Principal Ideal Theorem: If A is noetherian and $\mathfrak{p} \in \operatorname{Spec}(A)$ has height r, then there exists $a_1, \ldots, a_r \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime of $(a_1, \ldots, a_r)$.

4. Give an example of a prime ideal $\mathfrak{p}$ of height 1 in a noetherian ring A such that $\mathfrak{p}$ is not principal.

5. Prove that prime ideals in a noetherian ring satisfy the descending chain condition.

6. Determine the dimension of the ring $\mathbb{Z}[X]$ of polynomials in one variable with integer coefficients.

7. Suppose A is noetherian and I is any ideal of A. Show that $\dim A/I = \max\{\dim A/\mathfrak{p} : \mathfrak{p} \in \operatorname{Ass}(A/I)\} = \max\{\dim A/\mathfrak{p} : \mathfrak{p} \in \text{Min}(A/I)\}$.

8. Let Δ be a simplicial complex with vertex set $V = \{1, 2, \ldots, n\}$ and I_Δ be the ideal of $k[X_1, \ldots, X_n]$ as defined in Q. 14 of Problem Set 2. Consider the residue class ring $R_\Delta := k[X_1, \ldots, X_n]/I_\Delta$. Show that $\dim R_\Delta = d + 1$, where d is the (topological) dimension of Δ. [Note: R_Δ is called the *face ring* or the *Stanley-Reisner ring* associated to Δ.]

9. If a rational number satisfies a monic polynomial in $\mathbb{Z}[X]$, then show that it must be an integer. Deduce that $\mathbb{Z}$ is a normal domain. More generally, show that any UFD is a normal domain.

10. If B/A is an integral extension of rings, then show that B/J is integral over $A/J \cap A$ for every ideal J of A. Further, if S is a multiplicatively closed subset of A, then show that $S^{-1}B$ is an integral extension of $S^{-1}A$.

11. If A is a normal domain and S is a multiplicatively closed subset of A such that $0 \notin S$, then show that $S^{-1}A$ is a normal domain.

12. Show that if A is a domain, then A is normal if and only if $A[X]$ is normal.

13. If A is a normal domain, K is its quotient field, and x is an element of a field extension L of K such that x is integral over A, then show that the minimal polynomial of x over K has its coefficients in A.

14. Consider the subring $A = \mathbb{Z}[\sqrt{5}]$ of $\mathbb{C}$ and let $\alpha = (1 + \sqrt{5})/2$. Show that α is in the quotient field of A and α is integral over A, but $\alpha \notin A$.

15. Suppose k is an infinite field and $f \in k[X_1, \dots, X_n]$ is a nonzero polynomial. Prove that there exist some $a_1, \dots, a_n \in k$ such that $f(a_1, \dots, a_n) \neq 0$. Further show that if $n \geq 1$ and f is a nonconstant homogeneous polynomial in $k[X_1, \dots, X_n]$, then there are $c_2, \dots, c_n \in k$ such that $f(1, c_2, \dots, c_n) \neq 0$.

16. Prove the Tilting of Axes Lemma without the hypothesis that k is an infinite field by proceeding as follows. Given a nonconstant polynomial $f \in k[X_1, \dots, X_n]$, let e an integer greater than any of the exponents of $X_1, \dots, X_n$ appearing in f, and let $m_i = e^{i-1}$ for $2 \leq i \leq n$. Show that if $X_i' = X_i - X_1^{m_i}$ (instead of $X_i' = X_i - c_i X_1$), then $f = cX_1^m + g_1 X_1^{m-1} + \cdots + g_m$ for some $c \in k$ with $c \neq 0$ and $g_1, \dots, g_m \in k[X_2', \dots, X_n']$. Now argue as in the case of infinite k.

17. Let $A = k[X_1, \dots, X_n]$. Prove that $\operatorname{ht}(X_1, \dots, X_r) = r$ for $1 \leq r \leq n$.

18. Let k be a field and B be a domain and a f. g. algebra over k. Prove that the Krull dimension of B is equal to the transcendence degree of (the quotient field of) B over k.

19. A refined version of Noether's Normalization Lemma is as follows.

    Let $B = k[x_1, \dots, x_n]$ be a f. g. *algebra over a field k and $J_1 \subseteq \cdots \subseteq J_m$ be a chain of nonunit ideals of B. Then there exist $\theta_1, \dots, \theta_d \in B$ and nonnegative integers $r_1 \leq \cdots \leq r_m$ satisfying the following.*

    (i) $\theta_1, \dots, \theta_d$ *are algebraically independent over* k*,*

    (ii) B *is integral over* $A = k[\theta_1, \dots, \theta_d]$*; in particular,* B *is a finite* A*-module,*

    (iii) $J_i \cap A = (\theta_1, \dots, \theta_{r_i})A$ *for* $1 \leq i \leq m$.

    Assume this, and use it to show that if B is a domain and a f. g. algebra over a field k, then for any prime ideal P of B, we have $\dim B = \operatorname{ht} P + \dim B/P$.

20. Consider $B = k[X, Y, Z]/(XY, XZ) = k[x, y, z]$ and $\mathfrak{p} = (y, z)$. Show that $\dim B = 2$, whereas $\operatorname{ht} \mathfrak{p} = 0$ and $\dim B/\mathfrak{p} = 1$. Deduce that the last assertion in the previous problem is false if B is not a domain.