PROBLEMS

Section 1: Similarity Geometry and $\mathbb{C}$

1) Let (X, d) be a metric space.

Definition: A bijective map $f : X \to X$ is said to be an *isometry* if for all $P, Q \in X$, $d(f(P), f(Q)) = d(P, Q)$.

Definition: $I(X, d) = \{f : X \to X | f \textit{ is an isometry}\}$

Definition: A bijective map $f : X \to X$ is said to be a *similarity* if there exists $r > 0$, such that for all $P, Q \in X$, $d(f(P), f(Q)) = rd(P, Q)$.

The number r is called the *similarity factor* of f, and is denoted by $\sigma(f)$.

Definition: $S(X, d) = \{f : X \to X | f \textit{ is a similarity}\}$

Problem 1: The map $\sigma : S(X, d) \to \mathbb{R}_{>0}$ is a homomorphism of groups, and $ker\sigma = I(X, d)$.

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2) Let V be an n-dimensional vector space over $\mathbb{R}$, and Q a positive definite quadratic form on V. Let $d : V \times V \to \mathbb{R}_{\geq 0}$ be the distance defined by Q. Recall that $d(\vec{v}, \vec{w}) = \sqrt{Q(\vec{v} - \vec{w})}$. We denote the corresponding bilinear form by $<,>$.

Definition: The Euclidean n-space $\mathbf{E}^n$ is the pair (V, d).

Definition: $GL(V) = \{A : V \to V | A \textit{ is an invertible operator}\}$, is called the General Linear Group of V.

Definition: $Aff(V) = \{\alpha : V \to V | \alpha(\vec{v}) = A\vec{v} + \vec{b},\ A \in GL(V),\ \vec{b} \in V\}$, is called the Affine group of V.

Definition: $O(V, Q) = \{A \in GL(V) | \textit{for all } \vec{v}, \vec{w}, \in V, < A\vec{v}, A\vec{w} >=< \vec{v}, \vec{w} >\}$, is called the Orthogonal Group of (V, Q).

Remark: Notice that $GL(V), Aff(V), O(V, Q)$ are the full symmetry groups of three *structures* on the underlying set V, namely, the *linear structure*, the *affine structure*, and the *quadratic space structure*.

Problem 2: Show that $I(\mathbf{E}^n) = \{\vec{x} \mapsto A\vec{x} + \vec{b} | A \in O(V, Q), \vec{b} \in V\}$.
ii) $S(\mathbf{E}^n) = \{\vec{x} \mapsto rA\vec{x} + \vec{b} | r \in \mathbb{R}{>}\ 0, A \in O(V, Q), \vec{b} \in V\}$.

(*Hint*: i) Let $f \in I(\mathbf{E}^n)$, $f(\vec{0}) = \vec{b}$. Consider $\tau_{\vec{b}} : \vec{x} \to \vec{x} + \vec{b}$. Let $g = \tau_{\vec{b}}{}^{-1} o f$. So $g(\vec{0}) = \vec{0}$.
ii) Show that g preserves $<,>$.
iii) Show that g preserves vector addition and scalar multiplication.)

Problem 3: Show that $S(\mathbf{E}^n) = \{\vec{x} \mapsto rA\vec{x} + \vec{b} | r \in \mathbb{R}{>}\ 0, A \in O(V, Q), \vec{b} \in V\}$.

Remark: Notice that the problems 2 and 3 identify the full symmetry groups of the geometrically defined *Euclidean metric structure* and the *Euclidean similarity* *structure* on the underlying set V. This may be considered as the gist of the basic contribution of Descartes to Euclidean geometry.

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3) **Cartesian and Euclidean Co-ordinate Systems**: Let $\underline{\mathbf{e}} = (e_1, e_2, \ldots, e_n)$ be a *frame.* i.e. an ordered basis, of V. Any $\vec{v}$ in V can be expressed uniquely as $\vec{v} = \sum_{i=1}^{n} x_i \vec{e}_i$. This representation sets up an isomorphism $\phi : V \to \mathbb{R}^n$, which maps $\vec{e}_i$ onto the i-th standard basis vector (with i-th component 1, and rest 0) of $\mathbb{R}^n$. The map ϕ, or more loosely $\vec{x} = (x_1, x_2, \ldots, x_n)$, is called a *Cartesian co-ordinate system* on V centered at the origin. More generally, $(y_i = x_i - a_i)$ is called a *Cartesian co-ordinate system on* V *centered at* $\vec{a} = \sum_{i=1}^{n} a_i \vec{e}_i$.

Let V be equipped with a positive definite quadratic form Q. Let $\underline{\mathbf{e}} = (e_1, e_2, \ldots, e_n)$ be an orthonormal basis w.r.t Q. Then the corrsponding co-ordinate system is called an *Euclidean co-ordinate system* on V centered at the origin. Similarly we define *Euclidean co-ordinate system on* V *centered at* $\vec{a} = \sum_{i=1}^{n} a_i \vec{e}_i$.

Active and Passive Viewpoints: Fix a Cartesian co-ordinate system ϕ_o on V. Let α be an element of $Aff(V)$. Then obviously $\phi_o o \alpha$ is another Cartesian co-ordinate system. Moreover *every* Cartesian co-ordinate system is obtained this way, and the corresponding α is uniquely determined. This fact may be expressed in the language of group theory as follows. Let $\mathcal{C}$ be the set of all Cartesian co-ordinate systems. Then $Aff(V)$ acts on $\mathcal{C}$ and this action is *simply transitive.* Notice that $\mathcal{C}$ and $Aff(V)$ are in a bijective correspondence. However $Aff(V)$ has a distiguished element, namely the *identity,* but $\mathcal{C}$ has no such element. *After* one fixes an element in $\mathcal{C}$, one can canonically set up the bijective correspondence between $\mathcal{C}$ and $Aff(V)$. In the language of group theory, $\mathcal{C}$ is a *principal homogeneous space* of $Aff(V)$.

An element α in $Aff(V)$ may be considered as actually "moving points in V". This is the *active* viewpoint. On the other hand, $\phi_o o \alpha$ may be considered as "re-naming the points", or "the same person changing clothes". This is the *passive* viewpoint. Depending on the context, both viewpoints are important, although the underlying "formulas" are the same. Mixing the two views, one gets a remarkable conclusion that if α, β, γ are three elements in $Aff(V)$, and $\beta = \gamma o \alpha o \gamma^{-1}$, then the *dynamics of* α *and* β *are the same.* For if one confuses γ with $\phi_o o \gamma$ and considers it as change of co-ordinates then α and β are "the same person with different clothes". Otherwise they are different transformations, but their actions, for example their orbit structure, are the same. Thus the notion of conjugacy in group theory is a basic notion of *dynamic* origin.

Whatever we have said about $Aff(V)$ applies to all groups and their actions. We hope, that the reader will find that this viewpoint consciously applied to classical Complex Analysis also provides a valuable perspective on that subject.

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4) **Problem 4**: Show that

$$I(\mathbf{E}^2) \approx \{z \mapsto az + b | a, b \in \mathbb{C}, |a| = 1\} \cup \{z \mapsto a\bar{z} + b | a, b \in \mathbb{C}, |a| = 1\}.$$

Problem 5: Show that

$$S(\mathbf{E}^2) \approx \{z \mapsto az + b | a, b \in \mathbb{C}, a \neq 0\} \cup \{z \mapsto a\bar{z} + b | a, b \in \mathbb{C}, a \neq 0\}.$$

Problem 6: Let $f : \mathbb{C} \to \mathbb{C},\ f(z) = az + b,\ a \neq 0,\ or\ 1$.

i) Show that f has a unique fixed point.

ii) Call the fixed point z_o. Translate the coordinate system so that z_o is the origin, i.e. $z \to w = z - z_o$. Express f in terms of w, and show that dynamically f is a stretch-rotation, (which may degenerate to a stretch or a rotation, or identity.)

Problem 7: Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = a\bar{z} + b$, $a \neq 0$ and $|a| \neq 1$.

i) Show that f has a unique fixed point.

ii) Call the fixed point z_o. Translate the coordinate system so that z_o is the origin, i.e. $z \to w = z - z_o$. Express f in terms of w. Further rotate the sustem through $-\theta/2$ where $\arg a = \theta$, i.e $w \to u = e^{-i\theta/2}w$ and show that dynamically f is a stretch-reflection.

Problem 8: Let $f : \mathbb{C} \to \mathbb{C}$, $f(z) = a\bar{z} + b$, $|a| = 1$. Show that $f^2 = identity$ if and only if $a\bar{b} + b = 0$. Show that dynamically f is a reflection in a line or a glide-reflection.

Problem 9: Using complex numbers, prove the "Fundamental Theorem of Plane Euclidean Geometry".

Theorem: i) An orientation-preserving isometry of $\mathbf{E}^2$ is either the identity, a translation, or a rotation.

ii) An orientation-reversing isometry of $\mathbf{E}^2$ is either a reflection in a line or a glide-reflection.

iii) A non-isometric similarity of $\mathbf{E}^2$ is either a strech-rotation or a stretch-reflection according as it or preserves or reverses orientation.

Remark: The classical plane Euclidean geometry misses the orientation issues. For example, Euclid talked about *similar* triangles, but not *similarly oriented* ordered triangles. The proofs are often provided by appealing to figures, which is certainly a good idea for providing motivation. However it is often not clear whether the pictures cover all the cases of the asserted statement. On the other hand, the Cartesian proofs based on two real variables are often messy because it is difficult to deal with *oriented* angles. (Consider for example, the assertion that "the angle in a circular arc is constant". This is really a statement about about oriented angles, not just angular measures) *Complex numbers provide a neat calculus of oriented angles.* Making a choice between the two choices of $\sqrt{-1}$, calling it i, equating it with the point $(0,1)$ in $\mathbb{R}^2$, and setting $\arg i = \pi/2$ rather than $-\pi/2$, one builds in the *oriented* angles in the very structure.The following exercises are intended to sensitize the reader to some of these issues. Prove them using complex numbers.

Problem 10: Let γ be an arc from A to B of a circle with center O, so that the region bounded by the segments OA, γ, BO is counterclockwise oriented. Show that for any point C in γ, $A \neq C \neq B$, the counter-clock wise oriented angle $\angle BCA$ is constant.

[*Hint*: Using the two-point transitivity of $S(\mathbf{E}^2)$, we may choose the Euclidean coordinates so that O is the origin, $A = 1$, $B = e^{i\alpha}$, $C = e^{i\theta}$, where $0 < \theta < \alpha < 2\pi$. Then $\angle BCA$ is $\arg \frac{1-e^{i\theta}}{e^{i\alpha}-e^{i\theta}}$. Show that it is actually $\pi - \frac{1}{2}\alpha$. Note also that actually the argument of a complex number is defined only up to an additive integral multiple of 2π. So to interpret it as a real number, we need to choose the representative which lies in $(0, 2\pi)$.]

Problem 11: Let γ be an arc from A to B of a circle with center O, so that the region bounded by the segments OA, γ, BO is counterclockwise oriented. Let

γ' be the opposite arc from B to A. Show that for any points C in γ, $A \neq C \neq B$, and D in γ', $B \neq D \neq A$, the sum of the counter-clock wise oriented angles $\angle BCA + \angle ADB$ is constant.

Definition The *cross-ratio* of an ordered set of four distinct points $(z_1, z_2; z_3, z_4)$ in $\mathbb{C}$ is defined to be

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

Problem 12: Let $(z_1, z_2; z_3, z_4)$ be an ordered set of four distinct points in $\mathbb{C}$.

i) Show that they lie on a line or on a circle iff their cross-ratio is a (non-zero) real number.

ii) Show further that the sign of the cross-ratio is positive, resp. negative according as the four points lie in a linear (or cyclic) order, or do not lie in a linear (or cyclic) order.

[*Remark*: The statement ii) is actually equivalent to problems 10 and 11 combined.]

Problem 13: (Ptolemy's theorem, which generalizes the Bodhayana-Pythagoras theorem). Let A, B, C, D be four points in cyclic order lying on a circle. Show that

$$d(A, B)d(C, D) + d(A, D)d(C, B) = d(A, C)d(B, D).$$

(That is *for a cyclic quadrilateral, the sum of the products of the lengths of two pairs of opposite sides is equal to the product of the lengths of the diameters. For a rectangle the statement reduces to the Bodhayana-Pythagoras theorem.*)

[*Hint*: Use the identity

$$(z_1 - z_2)(z_3 - z_4) + (z_1 - z_3)(z_4 - z_2) + (z_1 - z_4)(z_2 - z_3) = 0.]$$

Problem 14: (Euler's Theorem) Let A, B, C be three non-collinear points in $\mathbf{E}^2$. Let A', B', C' be the midpoints of the line segments AB, BC, CA. Let D, E, F' be the feet of the perpendiculars from A, B, C on the (possibly extended) lines BC, CA, AB, respectively. Then A', B', C', D, E, F lie on the same circle.

[*Hint*: By symmetry, it suffices to prove A', B', C', D lie on a circle. Using the two-point transitivity of $S(\mathbf{E}^2)$, choose Euclidean co-ordinates so that D is the origin, B, C are on the $x-$ axis, and A has the z-coordinate i. Now compute an appropriate cross-ratio.]

Section 2: Hyperbolic Geometry and Schwarz Lemma
Notation:

$\mathbb{R}$ for the real numbers,
$\mathbb{C}$ for the complex numbers,
Δ for the open unit disk,
S^1 for the unit circle,
$\mathcal{U}$ for the upper half-plane,
$\mathcal{R}$ for the right half-plane,
$\widehat{\mathbb{C}}$ for the Riemann sphere
$\mathcal{A}(\mathbb{C}), \mathcal{A}(\Delta), \mathcal{A}(\mathcal{U}), \mathcal{A}(\mathcal{R}), \mathcal{A}(\widehat{\mathbb{C}})$ denote the respective holomorphic automorphism groups.

$\bar{\mathcal{A}}(\mathbb{C}), \bar{\mathcal{A}}(\Delta), \bar{\mathcal{A}}(\mathcal{U}), \bar{\mathcal{A}}(\mathcal{R}), \bar{\mathcal{A}}(\widehat{\mathbb{C}})$ denote the respective cosets of all anti-holomorphic (and orientation-reversing) maps.

Problem 1 (Cayley Transform) Show that i) $C : z \to \frac{z-i}{z+i}$ maps $\mathcal{U}$ bijectively onto Δ. ii) Find C^{-1}.

Problem 2 Consider

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}, \quad |a| < 1.$$

i) Show that ϕ_a maps Δ onto itself.
ii) Show that ${\phi_a}^{-1} = \phi_{-a}$.
iii) Let $a = re^{i\theta}$. Show that $e^{i\theta}$, resp. $-e^{i\theta}$, is the repelling, resp. attracting, fixed point of ϕ_a.
iv) Show that ϕ_a leaves the line containing $a, 0, -a$ invariant. More generally it leaves all circles passing through the fixed points of ϕ_a invariant.
v) Draw a schematic picture indicating the dynamics of ϕ_a.

Problem 3 Applying the Schwarz lemma show that all holomorphic automorphisms of Δ are of the form $z \mapsto e^{i\theta}\phi_a(z)$, where ϕ_a is as in Problem 2.

Remark: We have defined Δ in terms of a fixed coordinate z on $\mathbb{C}$. However it is sometimes convenient to think of z as one of the possible *hyperbolic coordinates* on Δ, and $\zeta_{\theta,a} = e^{i\theta}\phi_a(z)$, as defining another *hyperbolic coordinate* on Δ. This is again the interplay between the *active* and *passive* viewpoints which we encountered earlier, in linear, affine, or Euclidean geometries.

Even more generally, let X be a Riemann surface admitting a holomorphic isomorphism $\phi : X \to \Delta$. Then ϕ itself may be considered as defining a *hyperbolic coordinate* on X. It is by no means unique. In fact the Problem 3 shows that if $z = \phi(P)$ is one such coordinate then $e^{i\theta}\phi_a(z) = e^{i\theta}\phi_a(\phi(P))$ is also such a coordinate, and these are *all* such coordinates on X.

Problem 4 i) Show that the map C (cf Problem 1) is the unique holomorphic map mapping $\mathcal{U}$ bijectively onto Δ, such that $C(0) = -1$. ii) Find all bijective holomorphic maps of $\mathcal{U}$ onto Δ.

Remark: The existence of a map C shows that the sets Δ and $\mathcal{U}$, equipped with their *holomorphic structure*, that is as *Riemann surfaces*, are equivalent. Thus C is a *hyperbolic co-ordinate* on $\mathcal{U}$.

Problem 5 Find all bijective holomorphic maps of Δ onto $\mathcal{R}$.

Remark: The map $z \to \frac{z-1}{z+1}$ is an analogue of the *Cayley transform* from $\mathcal{R}$ to Δ. It may also be considered as a hyperbolic co-ordinate on $\mathcal{R}$.

Problem 6 Show that $\mathcal{A}(\Delta)$ can also be represented as

$$z \to \frac{az+b}{\bar{b}z+\bar{a}}$$

where $|a|^2 - |b|^2 > 0$.

Problem 7 Using Problem 6, and C (cf. Problem 1), show that $\mathcal{A}(\mathcal{U})$ can be represented as

$$z \to \frac{az+b}{cz+d}$$

where a, b, c, d are real and $ad - bc > 0$.

Problem 8 Find a representation of $\mathcal{A}(\mathcal{R})$ analogous to the representation of $\mathcal{A}(\Delta)$ and $\mathcal{A}(\mathcal{U})$ encountered above.

Problem 9 Show that
i) $\mathcal{A}(\mathcal{U})$ preserves the Riemannian metric $\frac{|dz|}{y}$.
ii) $\mathcal{A}(\mathcal{R})$ preserves the Riemannian metric $\frac{|dz|}{x}$.

Problem 10 Show that $\mathcal{A}(\Delta)$ preserves the Riemannian metric $\frac{2|dz|}{1-|z|^2}$.
(Note: The normalizing factor "2" ensures that the Riemannian curvature is constant -1, and not -2.)

Problem 11 Show that the Riemannian metric $\frac{|dz|}{y}$ on $\mathcal{U}$ is also preserved by the bijective anti-holomorphic maps

$$\bar{\mathcal{A}}(\mathcal{U}) = \{z \to \frac{a\bar{z}+b}{c\bar{z}+d} | a, b, c, d \; real, and \; ad - bc < 0\}.$$

$\bar{\mathcal{A}}(\mathcal{U})$ also consists of the orientation-reversing isometries, and $\mathcal{A}(\mathcal{U}) \cup \bar{\mathcal{A}}(\mathcal{U})$ is the full group of isometries of the Riemannian metric $\frac{|dz|}{y}$.

Problem 12 Show that the Riemannian metric $\frac{2|dz|}{1-|z|^2}$ on Δ is also preserved by the bijective anti-holomorphic maps

$$\bar{\mathcal{A}}(\Delta) = \{z \to \frac{a\bar{z}+b}{\bar{b}\bar{z}+\bar{a}} | |a|^2 - |b|^2 > 0\}.$$

$\bar{\mathcal{A}}(\Delta)$ also consists of the orientation-reversing isometries, and $\mathcal{A}(\Delta) \cup \bar{\mathcal{A}}(\Delta)$ is the full group of isometries of the Riemannian metric $\frac{2|dz|}{1-|z|^2}$.

Problem 13(Polar Decomposition) Compute $C^{-1}\phi_a C$ to see that it is represented by a 2×2 positive definite symmetric matrix.
Remark: The representation of $\mathcal{A}(\Delta)$ as in Problem 3 is essentially the polar decomposition of $PU(1,1)$ based at 0.

Problem 14 Following the last comment in Problem 13, construct the polar decomposition of $PGL^+(2,\mathbb{R})$ based at $i = \sqrt{-1}$.

Problem 15 For a nonsingular 2×2 matrix A with entries in $\mathbb{C}$ let

$$c(A) = \frac{(trace\ A)^2}{\det A}.$$

i) Show that for any non-zero complex number u we have $c(A) = c(uA)$.
ii) Show that for the 2×2 matrices which can be associated to the representations of $\mathcal{A}(\Delta), \mathcal{A}(\mathcal{U})$, and $\mathcal{A}(\mathcal{R})$, the invariant $c(A)$ always takes positive real values.
iii) (Dynamical types) Suppose A is a 2×2 matrix associated to a transformation in $\mathcal{A}(\Delta), \mathcal{A}(\mathcal{U})$, or $\mathcal{A}(\mathcal{R})$. Suppose that the transformation denoted by A is not identity. Then it is elliptic, parabolic or hyperbolic, according as $c(A)$ is < 4, $= 4$, or > 4.

Problem 16 Let $d(a,b)$ denote the hyperbolic distance in Δ computed w.r.t. the Riemannian metric $\frac{2|dz|}{1-|z|^2}$.
i) Show that

$$d(0,a) = \ln \frac{1+|a|}{1-|a|}, \; \tanh \frac{1}{2} d(0,a) = |a|.$$

ii) Let $p(a,b) = |\frac{a-b}{1-\bar{b}a}|$. Show that

$$d(a,b) = \ln \frac{1+p(a,b)}{1-p(a,b)}, \ \tanh \frac{1}{2} d(a,b) = p(a,b).$$

iii) Show that

$$\cosh^2 \frac{1}{2} d(0,a) = \frac{1}{1-|a|^2}, \ \cosh^2 \frac{1}{2} d(a,b) = \frac{|1-\bar{a}b|^2}{(1-|a|^2)(1-|b|^2)}.$$

The metric spheres and lines in Euclidean, spherical, and hyperbolic geometries

In any metric space (X,d) one may define the *metric ball with a center P in X and radius r*, namely $D_P(r) = \{Q \in X \mid d(P,Q) \leq r\}$. Its boundary, namely, $S_P(r) = \{Q \in X \mid d(P,Q) = r\}$ is the *metric sphere with center P and radius r*. It is more subtle to discuss *lines* in a general metric space, but there is a very good theory of *lines*, or as they are more commonly called the *geodesics*, in Riemannian manifolds. We shall not go into these topics here. But we shall consider the three main examples of these notions, which can be developed in less sophisticated, synthetic, terms. Namely, the Euclidean Geometry, Spherical Geometry, and Hyperbolic Geometry.

The Euclidean case is well-known. A striking feature in this case is that *the lines through a point P are orthogonal to the metric spheres centered at P.*

The Spherical case follows the suit. Here the model space is the unit sphere $\mathbf{S}^n$ in $\mathbf{E}^{n+1}, n \geq 2$. The metric spheres in $\mathbf{S}^n$ are the same as the round $n-1$-dimensional spheres in $\mathbf{S}^n$, that is the intersections of n-dimensional hyperplanes in $\mathbf{E}^{n+1}$ cutting $\mathbf{S}^n$ in more than one points. *Define* the *lines* in $\mathbf{S}^n$ to be the arcs of the great circles, namely the intersections of the 2-dimensional planes passing through the center of $\mathbf{S}^n$ in $\mathbf{E}^{n+1}$. The metric spheres in $\mathbf{S}^n$ are the same as the round $n-1$-dimensional spheres in $\mathbf{S}^n$, that is the intersections of n-dimensional hyperplanes in $\mathbf{E}^{n+1}$ cutting $\mathbf{S}^n$ in more than one points. Again it is not difficult to see that *the lines through a point P are orthogonal to the metric spheres centered at P.*

The Hyperbolic case also follows the suit. Here the model space is the unit ball Δ^n in $\mathbf{E}^n, n \geq 2$. Another model is a half space $\mathbf{H}^n$ which is one of the two components of $\mathbf{E}^n - P^{n-1}$, where P^{n-1} is an $n-1$-dimensional hyperplane. The metric spheres in Δ^n (or $\mathbf{H}^n$) are the same as the round $n-1$-dimensional spheres contained in Δ^n. *Define* the *lines* in Δ^n to be the arcs of the round circles orthogonal to the boundary of Δ^n. To define the lines in $\mathbf{H}^n$, the only modification that is needed is that we need to allow the (Euclidean) lines perpendicular to P^{n-1} also as lines in $\mathbf{H}^n$. Again one may show that *the lines through a point P are orthogonal to the metric spheres centered at P.*

It is a very interesting point that *the metric spheres and the lines in hyperbolic geometry can be constructed using Inversive geometry, which is based on the notion of angle but not of distance.* These constructions actually identify the metric spheres and the lines as subsets of Δ^n (or $\mathbf{H}^n$). Their metric properties can be derived as an after-thought! This appears to be indeed the thought process of Poincaré which led him to "see" the presence of hyperbolic geometry in Complex Analysis of one variable.

Let P be in Δ^n (or $\mathbf{H}^n$). If P is the Euclidean center of Δ^n then the Euclidean metric spheres centered at P are also the hyperbolic metric spheres centered at P. Moreover the lines through P are the segments of the Euclidean lines through P contained in Δ^n. (There is no such point in the $\mathbf{H}^n$-model which is distinguished from the Euclidean perspective.)

Now let P be any point in Δ^n *different from the Euclidean center of* Δ^n. (In the $\mathbf{H}^n$ model let P be any point.) Let P^* be the inverse of P w.r.t the boundary of Δ^n. (In the $\mathbf{H}^n$ model let P^* be the mirror reflection of P in the bounding P^{n-1}.) Then the lines through P are just the portions of the round circles through P and P^* which are contained in Δ^n (or $\mathbf{H}^n$). The metric spheres with center P are just those $n-1$-dimensional spheres which cut the hyperbolic lines through P orthogonally. They may be constructed as follows. Let l be the Euclidean line through P and P^*. Let Q be any point on l. Let Q^* be a point on l such that the cross-ratio $\{P, P^*; Q, Q^*\} = -1$. Then the Euclidean sphere for which Q and Q^* are diametrically opposite points is also a metric sphere in hyperbolic geometry with center P and passing through Q. All metric spheres are obtained in this way.

The following three exercises will acquaint the reader with the basic pictures of metric spheres and lines in the 2-dimensional case with the models Δ, $\mathcal{U}$, and $\mathcal{R}$. We have already defined the lines above. Using some of the italicized facts in the above paragraphs verify the result in the following problem. For convenience we shall call 1-dimensional metric sphere, a *metric circle*.

Problem 17 i) Let $z_o = x_o + iy_o$ be in $\mathcal{U}$. Show that the metric circle with center z_0 and passing through a point $z_1 = x_o + iy_1$ is the Euclidean circle having z_1 and $z_2 = x_o + iy_2$ as diametrically opposite points where $y_1 y_2 = y_o^2$.

ii) Show that all lines through z_o are the portions of the circles passing through $z_o, \bar{z}_o$ and contained in $\mathcal{U}$

Problem 18 Describe the metric circles and lines in $\mathcal{R}$.

Problem 19 Describe the metric circles and lines in Δ.

Problem 20 Recall the standard form of *Schwarz's Lemma*: *Let* $f : \Delta \to \Delta$ *be a holomorphic function, such that* $f(0) = 0$. *Show that* $|f(z)| \leq |z|$, *with equality at any point iff* $f(z) = az$ *with* $|a| = 1$.

Let $f : \Delta \to \Delta$ be a holomorphic function, such that $f(0) = 0$. Let d denote the hyperbolic distance as in Problem 16. Show that $d(0, f(z)) \leq d(0, z)$, with equality at any point iff $f(z) = az$ with $|a| = 1$.

Problem 21 (Schwarz-Pick Lemma) Let $f : \Delta \to \Delta$ be a holomorphic function. Let d denote the hyperbolic distance as in Problem 16.

i) Show that for any holomorphic function $f : \Delta \to \Delta$,

$$\Big|\frac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}\Big| \leq \Big|\frac{z - w}{1 - \overline{w}z}\Big|$$

for all z, w in Δ, with equality at any point iff f is in $\mathcal{A}(\Delta)$. Note that in the notation of Problem 16, the inequality reads

$$p(f(z), f(w)) \leq p(z, w)$$

with equality at any point iff f is in $\mathcal{A}(\Delta)$.

[Hint: Consider w as fixed, and let $g(z) = \phi_{f(w)} o f o {\phi_w}^{-1}$. Then $g(0) = 0$.]

ii) Show that $d(f(z), f(w)) \leq d(z, w)$, with equality at any point iff f is in $\mathcal{A}(\Delta)$.

[Hint: The function tanh is strictly increasing.]

Problem 22 Show that for any holomorphic function $f : \Delta \to \Delta$,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}$$

for all z in Δ, with equality at any point iff f is in $\mathcal{A}(\Delta)$.

Problem 23 Let $f : \Delta \to \Delta$ be a holomorphic function with fixed point a in Δ. Show that

$$\Big|\frac{f(z)-a}{1-\overline{a}f(z)}\Big| \le \Big|\frac{z-a}{1-\overline{a}z}\Big|$$

If there is equality at any $z_o \neq a$ then f must be an elliptic automorphism.

Remark: This statement of the Schwarz Lemma reads: *For any hyprbolic coordinate $\zeta_{\theta,a} = e^{i\theta}\phi_a(z)$, centered at a, and any $f : \Delta \to \Delta$ such that $f(a) = a$ we have $|\zeta_{\theta,a}(f(z))| \le |\zeta_{\theta,a}(z)|$.*

Problem 24 Let X be a Riemann surface admitting a holomorphic isomorphism $\phi : X \to \Delta$. Let $f : X \to \Delta$ be an arbitrary holomorphic function. Assume that there exists a point P in X such that $\phi(P) = f(P)$. Then show that for all Q in X, we have $|f(Q)| \le |\phi(Q)|$.

Problem 25 If $f(z)$ is holomorphic and $\text{Im}f(z) \ge 0$ for $\text{Im}(z) > 0$, show that

$$\frac{|f(z)-f(z_0)|}{|f(z)-\overline{f(z_0)}|} \le \frac{|z-z_0|}{|z-\overline{z_0}|}.$$

with equality at one point iff f is in $\mathcal{A}(\mathcal{R})$.

Problem 26 i) Suppose f is a holomorphic automorphism of Δ such that f has two fixed points. Show that f must be the identity.

ii) Let $f : \bar{\Delta} \to \bar{\Delta}$ be a continuous function which is holomorphic in Δ. Suppose that it is *not* an automorphism of Δ. By Brouwer's theorem f has fixed point. Suppose moreover that a fixed point lies in Δ. Then show that the fixed point is unique.

Problem 27 If $f : \mathcal{R} \to \mathcal{R}$ is holomorphic and $f(1) = 1$ show that (i) $|f'(1)| \le 1$ and (ii) $|\frac{f(z)-1}{f(z)+1}| \le |\frac{z-1}{z+1}|$ for all z in $\mathbb{H}$.

Problem 28 Let $X \subset \mathbb{C}$ be an open set admitting a holomorphic isomorphism $\phi : X \to \Delta$. Let $f : X \to X$ be an arbitrary holomorphic function. Assume that there exists a point P in X such that $f(P) = P$. Then show that i) $|f'(P)| \le 1$, ii) for all Q in X, we have $|\phi o f(Q)| \le |\phi(Q)|$.

Problem 29 Does there exist a holomorphic function $f : \Delta \to \Delta$ such that $f(1/2) = 3/4$ and $f'(1/2) = 2/3$?

Problem 30 Is there a holomorphic function f on Δ such that $|f(z)| < 1$ for $|z| < 1$, $f(0) = 1/2$ and $f'(0) = 3/4$? If so, find such an f. Is it unique?

Problem 31 Suppose $|f(z)| \le 1$ for all $z < 1$ and f is holomorphic. If $f(0) = a$ show that

$$\frac{|f(0)|-|z|}{1-|f(0)||z|} \le |f(z)| \le \frac{|f(0)|+|z|}{1+|f(0)||z|}$$

for all $|z| < 1$.

Problem 32 Suppose $f : \Delta \to \mathbb{C}$ is a non-constant holomorphic function satisfying $\mathrm{Re} f(z) \geq 0$ for all z in Δ.

(a) Show that $\mathrm{Re} f(z) > 0$ for all z in Δ.

(b) If $f(0) = 1$ show that

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}$$

for all $|z| < 1$.

(c) Show that f also satisfies

$$|f(z)| \geq \frac{1 - |z|}{1 + |z|}$$

for all $|z| < 1$.

Section 3: Inversive Geometry, Stereographic Projection, and Möbius Transformations

1) **Problem 1** Let $\hat{\mathbf{E}}^n$ be the one-point compactification of $\mathbf{E}^n$. Show that $\hat{\mathbf{E}}^n$ is homeomorphic to $\mathbf{S}^n$.

[*Facts* i) The conformal structure on $\mathbf{E}^n$ extends to $\hat{\mathbf{E}}^n$. The extended conformal structure on $\hat{\mathbf{E}}^n$ is isomorphic to the standard conformal structure on a round sphere $\mathbf{S}^n$ in $\mathbf{E}^{n+1}$.

ii) For $n = 2$ equip $\mathbf{E}^2$ with an orientation. Notice that the choice of an orientation in $\mathbf{E}^2$ is equivalent to the choice of one of the two square roots of -1. Then we may identify the conformal structure on $\mathbf{E}^2$ with the holomorphic structure with $\mathbb{C}$. This holomorphic structure extends to a holomorphic structure on $\hat{\mathbf{E}}^2$. The space $\hat{\mathbf{E}}^2$ with its holomorphic structure is the *Riemann sphere*.]

Definition i) Let $\Sigma_{P,r}$ be a round sphere in $\mathbf{E}^n$ with center P and radius r. Define the map $f : \mathbf{E}^n - \{P\} \to \mathbf{E}^n - \{P\}$ as follows. Let Q be a point in $\mathbf{E}^n - \{P\}$. On a ray emanating from P and passing through Q choose Q^* such that $d(P,Q)d(P,Q^*) = r^2$. Then set $f(Q) = Q^*$.

ii) Let $\hat{\mathbf{E}}^n = \mathbf{E}^n \cup \{\infty\}$. Extend f further to $\hat{f} : \hat{\mathbf{E}}^n \to \hat{\mathbf{E}}^n$ by setting $\hat{f}(P) = \infty$, and $\hat{f}(\infty) = P$.

The map $\hat{f}$ is called the *inversion* in $\Sigma_{P,r}$, and is denoted by $\sigma_{\Sigma_{P,r}}$ or just $\sigma_{P,r}$.

Problem 2 Theorem The inversion $\sigma = \sigma_{P,r}$ has the following properties. Let $\Pi = \Pi^{n-1}$ denote an $n-1$-dimensional hyperplane and $\Sigma = \Sigma^{n-1}$ a round sphere in $\mathbf{E}^n$. Let $\hat{\Pi} = \Pi \cup \{\infty\}$.

i) σ fixes $\mathbf{S}^{n-1}$ pointwise, and $\sigma^2 = \textit{identity}$. It interchanges the two components of $\hat{\mathbf{E}}^n - \mathbf{S}^{n-1}$.

ii) If P is in Π then σ maps $\hat{\Pi}$ to itself.

iii) If P is not in Π then $\sigma(\hat{\Pi})$ is a Σ passing through P. The line passing through P and diametrically opposite point to P on Σ is perpendicular to Π.

iv) If P is in Σ then $\sigma(\Sigma)$ is a $\hat{\Pi}$ not passing through P. The line passing through P and diametrically opposite point to P on Σ is perpendicular to Π.

v) If P is not in Σ then $\sigma(\Sigma)$ is a Σ not passing through P.

Definition i) Let $\mathbf{S}^{n-1}$ be a round sphere in $\mathbf{E}^n$ and P a point in $\mathbf{S}^{n-1}$. Let τ be the tangent plane to $\mathbf{S}^{n-1}$ at P. Let Π be any $n-1$-dimensional hyperplane *parallel* to τ but *different* from τ. Define the map $f : \mathbf{S}^{n-1} - \{P\} \to \Pi$ as follows.

Let Q be a point in $\mathbf{S}^{n-1} - \{P\}$. On a line joining P to Q choose Q^* lying on Π. Then set $f(Q) = Q^*$.

ii) Let $\hat{\Pi} = \Pi \cup \{\infty\}$. Extend f further to $\hat{f} : \mathbf{S}^{n-1} \to \hat{\Pi}$ by setting $\hat{f}(P) = \infty$.

The map $\hat{f}$ is called the *stereographic projection* from $\mathbf{S}^{n-1}$ to $\hat{\Pi}$. It is denoted by $\pi_{\mathbf{S}^{n-1},P,\Pi}$, or for short by π, if other entities are understood.

Problem 3 Theorem The *stereographic projection* $\pi = \pi_{\mathbf{S}^{n-1},P,\pi}$ has the following properties. If H is a $n-2$-dimensional hyperplane in Π, then we denote $\hat{H} = H \cup \{\infty\}$ as a subset of $\hat{\Pi}$.

i) π maps a round $\mathbf{S}^{n-2}$ passing through P onto a $\hat{H} \subset \hat{\Pi}$.

ii) π maps a round $\mathbf{S}^{n-2}$ not passing through P onto a round $n-2$-dimensional sphere in $\hat{\Pi}$.

[*Hint* Let $\Sigma = \Sigma_{(P,s)}$ be a $n-1$-dimensional round sphere with center P and radius s, so chosen that σ_Σ maps $\mathbf{S}^{n-1}$ onto Π. (For example, if we take Π to be tangent to $\mathbf{S}^{n-1}$ at a point diametrically opposite to P, then we may take $s = 2r$ where r is the radius of $\mathbf{S}^{n-1}$. Then $\Sigma_{(P,s)|_{\mathbf{S}^{n-1}} = \pi}$.]

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2) **Definition** The Möbius group $\mathcal{M}(n)$ is the group acting on $\mathbf{S}^n$, where $\mathbf{S}^n$ is considered as $\hat{\mathbf{E}}^n$. It is generated by the inversions in the round Σ^{n-1}'s in $\mathbf{S}^n$.

[*Facts*: i) $\mathcal{M}(n)$ is the full group of conformal transformations of $\mathbf{S}^n$. It has two components, one consisting of orientation-preserving transformations, and the other by orientation-reversing transformations.

ii) It is isomorphic to the Lie group which consists of two out of the four components of $O(n+1,1)$.]

iii) Let $n = 2$, and identify $\hat{\mathbf{E}}^2$ with the Riemann sphere. Then the orientation-preserving subgroup of $\mathcal{M}(2)$ may be identified with

$$\{z \to \frac{az+b}{cz+d}| \;\; a,b,c,d \; in \; \mathbb{C} \; and \; ad-bc \neq 0\}.$$

The coset of orientation-reversing transformations may be identified with

$$\{z \to \frac{a\bar{z}+b}{c\bar{z}+d}| \;\; a,b,c,d \; in \; \mathbb{C} \; and \; ad-bc \neq 0\}.]$$

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3) **Dynamical types of elements in $\mathcal{M}(2)$.**

See Theorem 7.2 in the Notes on Mobius Geometry.

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4) **Steiner Pencils**:

Definition Let P, Q be two distinct points in $\hat{\mathbf{E}}^2$. The set $\mathcal{E}_{P,Q}$ of all round circles passing through P, Q is called an *elliptic Steiner Pencil.*

Definition The set of round circles orthogonal to $\mathcal{E}_{P,Q}$ is called a *hyperbolic Steiner Pencil.* It is denoted by $\mathcal{H}_{P,Q}$

Definition Let P be in $\hat{\mathbf{E}}^2$, and γ a circle passing through P. The family of round circles passing through P and tangent to C at P is called a *parabolic Steiner Pencil.* It is denoted by $\mathcal{P}_{P,\gamma}$

Problem 4 Let P be in $\mathbf{E}^2$. Consider the set $\mathcal{E}_{P,\infty}$ of (Euclidean) lines through P. Let $\mathcal{H}_{P,\infty}$ be the set of all circles with center P. Show that these two sets may be considered as elliptic and hyperbolic Steiner pencils, and every elliptic resp hyperbolic Steiner pencil is obtained as a transform of $\mathcal{E}_{P,\infty}$ resp. $\mathcal{H}_{P,\infty}$ by a suitable Möbius transformation, that is an element of $\mathcal{M}(2)$.

Problem 5 Let f be an element of $\mathcal{M}(2)$ which preserves orientation. Suppose that it has two distinct fixed points P, Q. Show that

i) f is elliptic (a rotation) iff it leaves every element of $\mathcal{E}_{P,Q}$ invariant, whereas it permutes the elements of $\mathcal{H}_{P,Q}$.

ii) f is hyperbolic (a stretch) iff it leaves every element of $\mathcal{H}_{P,Q}$ invariant, whereas it permutes the elements of $\mathcal{E}_{P,Q}$.

iii) f is loxo-dromic (a stretch-rotation) iff it leaves invariant $\mathcal{E}_{P,Q}$ and $\mathcal{H}_{P,Q}$ as sets (but no element in either set is left invariant.

Problem 6 Let f be an element of $\mathcal{M}(2)$ which preserves orientation. Suppose that it has exactly one fixed point P. Show that f leaves invariant $\mathcal{P}_{P,\gamma}$ for some circle γ through P.

Function Theory

We have distributed copies of the English translation of Riemann's thesis (1851), *Foundations for a general theory of functions of a complex variable.* We highly recommend its reading. Riemann's account is intuitive from modern perspective. Indeed, in Riemann's times there was no *Topology* as a field of mathematics. There was no notion of a *manifold.* Nor the notion of *simple connectivity* as we underatnd it today. However the roots of many developments in the last 150 years are found in this paper. In particular, one finds here

i) The very idea of a function and the dynamic viewpoint,

ii) Importance of the topological notion of simple connectivity, developed only in the context of surfaces,

iii) Classification of simply connected surfaces, and classification of compact connected orientable surfaces with boundary,

iv) Covering space theory. (Riemann does not distinguish between a "local homeomorphism" and a "covering space". In its proper generality the theory was developed by Poincaré in 1890's)

v) Complex holomorphic structures on surfaces, and one of the main steps towards their classification, namely the *Riemann mapping theorem.*

For a modern perspective, one should also add the input of the theory from *Lie groups.* Recognizing the roles of the Lie group $PSL_2(\mathbb{C})$ and its basic subgroups $PSU_2, PSU(1,1), andS(bfE^2)$ in the Complex Analysis of one variable, brings in great clarity. Under the influence of Felix Klein, Lie developed this field starting in 1880's.

Dynamic Viewpoint and Covering Space Theory: By a *surface* we mean a 2-dimensional, Hausdorff manifold, with a countable basis for topology.

A Basic Theorem of 2-dimensional Topology: *There are only two connected. simply connected surfaces up to homeomorphism, namely* $\mathbb{R}^2$ *and* $\mathbb{S}^2$.

A Basic Theorem of Complex Analysis: (Riemann-Koebe Uniformization theorem): *Up to equivalence of complex holomorphic structures there are only two such structures on* $\mathbb{R}^2$*, namely* Δ *and* $\mathbb{C}$*, and there is only one such structure on* $\mathbb{S}^2$*, namely the Riemann sphere.*

Definiton: A surface equipped with a complex holomorphic structure is called a *Riemann surface*.

Let X, Y be connected Riemann surfaces, and $f : X \to Y$ a holomorphic map. Then by the covering space theory, their simply connected coverings are Δ or $\mathbb{C}$, or $\mathbb{S}^2$. Then f lifts to a holomorphic map $\tilde{f}$ between two of these three objects.

Basically for topological reasons there is no non-constant holomorphic map from $\mathbb{S}^2$ to Δ or $\mathbb{C}$. By *Liouville's theorem*, there is no non-constant holomorphic map from $\mathbb{C}$ to Δ . The holomorphic maps from $\mathbb{S}^2$ to itself are just the rational functions $\mathbb{C}(z)$. Due to these remarks one sees the importance of studying the following topics.

i) $\mathbb{C}(z)$ and the dynamics of rational functions.
ii) the holomorphic maps from $\mathbb{C}$ to itself (*the entire funtions*), and their dynamics
iii) the holomorphic maps from $\mathbb{C}$ to $\mathbb{S}^2$ (*the entire meromorphic functions*) and their partial dynamics.
iv) the holomorphic maps from Δ to itself and their dynamics,
v) the holomorphic maps from Δ to $\mathbb{C}$, or to $\mathbb{S}^2$, and their partial dynamics.

Dynamic behavior at the fixed points; Let X be a Riemann surface, $f : X \to X$ so that f has a fixed point at P. Then $f'(P)$ has a meaning independent of the choice of a holomorphic co-ordianate at P. We call P an *expanding*, resp. *attracting*, resp. *neutral* fixed point according as $|f'(P)|$ is 10 resp. < 1, resp. $= 1$.

Problem 1 Let f be an orientation-preserving Möbius transformation of $\mathbb{S}^2$. Then it has at least one and at most two fixed points. Let P be a fixed point of f. Then
i) f is elliptic iff $|f'(P)| = 1$, and $f'(P) \neq 1$.
ii) f is hyperbolic iff $f'(P) \in \mathbb{R}_{>0}$, and $f'(P) \neq 1$.
iii) f is parabolic iff $f'(P) = 1$, and $f \neq$ *identity*.
iv) f is loxo-dromic iff $f'(P) \notin \mathbb{R}$ and $|f'(P)| \neq 1$.

[*Remark*: Notice that this is a very different characterization than the algebraic one in terms of traces and determinants, or the geometric one in terms of Steiner pencils.]

Problem 2 Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, given by $f(z) = \frac{1}{2}(z + \frac{1}{z})$.
Show that f has three fixed points, two of them attracting, and the third repelling.

Riemann Maps:

Problem 1 Find a bijective holomorphic map of the half-strip $-\frac{1}{2} \leq \text{Re}z \leq \frac{1}{2}$, $\text{Im}z \geq 0$ onto the open unit disk.

Problem 2 Find a bijective holomorphic map of the intersection of $|z| \leq 1$ and $|z-1| \leq 1$ onto the open unit disk.

Problem 3 Find a bijective holomorphic map of the intersections of the closed regions $vertz| \leq 1$ and $|z - 1/2| \geq 1/2$ onto the open unit disk.

Problem 4 Find a bijective holomorphic map from the strip $U = \{x + iy : 0 < y < \pi\}$ onto the half-strip $V = \{x + iy : 0 < y < \pi/2, 0 < x < \infty\}$.

Problem 5 Let f be holomorphic for $|z| < 1$ and let $D(r, f)$ denote the image of $|z| < r$ under f where $0 < r < 1$. Suppose also that f is injective for $|z| < r$ and $A(r, f)$ is the area of the set $D(r, f)$. Show that

$$A(r, f) = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$$

. Deduce the least value of $A(r, f)$ (for fixed r and a_1.)

Proper holomorphic maps

Problem 1 Let f be entire and proper; show that f must be a polynomial.

Problem 2. Prove that if f is an entire function with a pole at infinity of order m, then it must be a polynomial of degree m.

Problem 3 Prove that any function which is meromorphic in the extended plane is rational, and conversely.

Problem 4 Show that there exists one and only one function f holomorphic on Δ, continuous on $\overline{\Delta}$, such that $f(1) = 1$, $f(S^1) \subset S^1$ and f has finitely many zeroes at given points of Δ with given orders.

Problem 5 Let f be an entire function and suppose that there exists a constant M, and $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree $\leq n$.

Problem 6 If f is an entire function and $|f(z)| \leq C|z|^n$ for all z in $\mathbb{C}$, then show that $f(z) = Cz^n$.

Problem 7 Let p be a nonconstant polynomial and show that each component of $\{z : |p(z)| < c\}$ contains a zero of p.

Miscellaneous

Problem 1 Let G be a region and suppose that $f : G \to \mathbb{C}$ is holomorphic and $a \in G$ such that $|f(a)| \leq |f(z)|$ for all z in G. Show that either $f(a) = 0$ or f is a constant.

Problem 2 Show that an entire function whose range is contained in $\mathbb{C} \setminus \{t \in \mathbb{R} : t \geq 0\}$ must be constant.

Problem 3 Let f be a non-constant entire function; show that the range of f is dense in $\mathbb{C}$.

Problem 4 Let f be holomorphic and injective in $\{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that 0 cannot be an essential singularity of f.

Problem 5 Show that all holomorphic automorphisms of $\mathbb{C}$ are of the form $z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0$.

Problem 6 If $e^{f(z)} + e^{g(z)} = 1$ for all z, where f and g are entire functions, show that f and g are constants.

Problem 7 Let $f : \Delta \to \Delta$ and $g : \Delta \to \Delta$ be holomorphic and $f(0) = g(0)$. Suppose $f(\Delta) \subset g(\Delta)$ and g is injective. Show that $|f'(0)| \leq |g'(0)|$.

Problem 8 If F is a nonconstant holomorphic function on a bounded open set G and is continuous on the closure $\overline{G}$, show that either f has a zero in G or $|f|$ assumes its minimum value on ∂G.

Problem 9 Let G be a bounded region and suppose f is continuous on $\overline{G}$ and holomorphic in G. Show that if there exists a constant $c \geq 0$ such that $|f(z)| = c$ for all $z \in \partial G$, then either f is a constant function or f has a zero in G.

Problem 10 Let f be continuous on a bounded region $\overline{\Omega}$ and holomorphic in Ω. Show that if $\mathrm{Re}(f)$ vanishes on the boundary of Ω then f must be a constant.

Problem 11 Suppose that f is holomorphic on a bounded region $\Omega \subset \mathbb{C}$ and assume that $\mathrm{Im}(f)$ extends to a continuous function on $\overline{\Omega}$, that vanishes on $\partial\Omega$. Show that f is a constant real number.

Problem 12 Let f be holomorphic on a region Ω such that $\mathrm{Re}(f)$ has a local minimum inside Ω. Show that f must be constant.

Problem 13 Let B be the open unit ball

$$B = \{(z, w) : |z|^2 + |w|^2 < 1\}$$

in $\mathbb{C}^2$. Show that there cannot be any nonzero holomorphic function on Δ whose graph is contained in B.

Problem 14 Let f be an entire function such that $\mathrm{Re}f(z) \geq -1$ for all $z \in \mathbb{C}$. Show that f must be a constant.

Problem 15 Let f be holomorphic in the disk $B(0; R)$ and for $0 \leq r < R$ define $A(r) = \max\{\mathrm{Re}\ f(z) : |z| = r\}$. Show that unless f is a constant, $A(r)$ is a strictly increasing function of r.

Problem 16 Suppose f is holomorphic in Δ such that $|f(z^2)| \geq |f(z)|$ for all $z \in \Delta$. Show that f is a constant.

Problem 17 Given $R > 0$ and $z_0 \in \mathbb{C}$, let $B = \{z \in \mathbb{C} : |z - z_0| \leq R\}$. Let dA = area measure on B. Show that if f is holomorphic in a neighborhood of B, then

$$f(z_0) = \frac{1}{\pi R^2} \int_B f(z) dA.$$

Problem 18 With the same conditions of Problem 18, suppose $L(r, f)$ is the length of the boundary of $D(r, f)$. Show that $L(r, f) \geq 2\pi r|f'(0)|$.

Problem 19 Show that if f is holomorphic on an open set containing the disk $\overline{B(a,R)}$ then

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a+re^{i\theta})|^2 dr d\theta.$$

Problem 20 Let f be an entire function in the plane and suppose that there exists a constant M such that

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq M$$

for $r \geq 0$. Show that f must be a constant.

Problem 21 Let f and g be entire functions with $|f(z)| \leq |g(z)|$ for every $z \in \mathbb{C}$. Show that there exists a constant K such that $f(z) = Kg(z)$.

Problem 22 Suppose f is holomorphic in $\Omega \setminus \{a\}$ for some region Ω, $a \in \Omega$ and f is bounded in $B = \{z : 0 < |z-a| < R\}$ for some $r > 0$. Prove that f has a removable singularity at a.

Problem 23 Suppose a is a pole for f; show that a is an essential singularity for e^f.

Problem 24 Prove that an isolated singularity of $f(z)$ is removable as soon as $\mathrm{Re}f(z)$ or $\mathrm{Im}f(z)$ is bounded above or below.

Problem 25 Let f be holomorphic in $G = \{z : 0 < |z-a| < r\}$ except that there is a sequence of poles $\{a_n\}$ in G with $a_n \to a$. Show that for any w in $\mathbb{C}$, there exists a sequence $\{z_n\}$ in G with $\lim z_n = a$ and $\lim f(z_n) = w$.

Problem 26 Suppose $f : \Delta \setminus \{0\} \to \mathbb{C}$ is holomorphic and assume that

$$\int_{0<|x+iy|<1} |f(x+iy)|^2 dx dy < \infty.$$

Prove that f can be extended uniquely to a holomorphic function on Δ.

Problem 27 Let G be a region and $H(G)$ denote the set of all holomorphic functions on G and $M(G)$ the set of all meromorphic functions on G. Show that:
(i) $H(G)$ is an integral domain, and
(ii) $M(G)$ is a field.

Problem 28 Let $\{f_n\} \subset H(G)$ be a sequence of 1-1 functions which converge to f. Show that either f is 1-1 or f is a constant function.

Problem 29 Prove that if G is a region and $\{f_n\} \subset H(G)$ is locally bounded and $f \in H(G)$ has the property that $A = \{z \in G : \lim f_n(z) = f(z)\}$ has a limit point in G, then $f_n \to f$ compactly.

Problem 30 Show that if $\mathcal{F} \subset H(G)$ is normal, then $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also normal.

Problem 31 Suppose $\mathcal{F}$ is normal in $H(G)$ and Ω is open in $\mathbb{C}$ such that $f(G) \subset \Omega$ for every f in $\mathcal{F}$. Show that if g is holomorphic on Ω and is bounded on bounded sets, then $\{g \circ f : f \in \mathcal{F}\}$ is normal.

Problem 32 Let G be a region and K a non-discrete subset of G. Suppose that $f, g : G \to \mathbb{C}$ are holomorphic without zeroes and satisfy

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{g'(\zeta)}{g(\zeta)}$$

for all $\zeta \in K$. Find a relation between f and g.

Problem 33 i) Given $R > 0$ and $z_0 \in \mathbb{C}$, let $B = \{z \in \mathbb{C} : |z - z_0| \le R\}$. Let $dA =$ area measure on B. Show that if f is holomorphic in a neighborhood of B, then

$$f(z_0) = \frac{1}{\pi R^2} \int_B f(z) dA.$$

ii) With the same conditions as in i), suppose $L(r, f)$ is the length of the boundary of $D(r, f)$. Show that $L(r, f) \ge 2\pi r |f'(0)|$.

Problem 34 Let f be holomorphic on $\mathbb{C}$ except for isolated singularities at $a_1, ..., a_m$. Show that $\text{Res}(f; \infty) = -\sum_{k=1}^{m} \text{Res}(f; a_k)$.

Problem 35 Let p be a nonconstant polynomial and show that each component of $\{z : |p(z)| < c\}$ contains a zero of p.

Problem 36 Let f be holomorphic on $\overline{B(0; R)}$ and let $M(r) = \max\{|f(z)| : |z| = r\}$, and $A(r) = \max\{\text{Re } f(z) : |z| = r\}$. Prove that for $0 < r < R$, if $A(r) \ge 0$,

$$M(r) \le \frac{R + r}{R - r} \Big(A(R) + |f(0)| \Big).$$