Regional Mathematical Olympiad- 2006

Maharashtra and Goa Region

17th December 2006

Max. Marks: 100 **Time : 4 hours**

N.B.(i) There are 8 questions. All questions are compulsory.

(ii) Mathematical reasoning will be taken into consideration while assessing the answer.

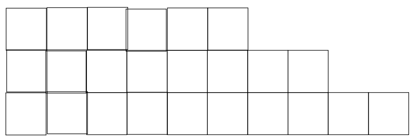
(iii) Figures to the right indicate full marks for the question.

1. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+c)(b+a)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}.$$ **[10]**

2. Find all positive integers n such that the number $n(2^{n-1}) + 1$ is a perfect square. **[10]**

3. In how many ways can 7 X's be written so that each unit square contains at most one X and no row is empty in the following figure? **[10]**



4. Let PA be a common chord of circles C_1 and C_2. Extend PA to Q such that A is midpoint of PQ. Let the tangent to the circle C_1 drawn at P intersect C_2 at R and the tangent to the circle C_2 drawn at P intersect C_1 at S. Show that P, Q, R, S are concyclic. **[12]**

5. Let $\triangle ABC$ be a triangle with $\angle B$ as an obtuse angle and $\angle A < 60°$. Let P be a point on the side AB such that $\angle CPB = 60°$. Let D be the point on CP which also lies on the internal angle bisector of $\angle A$. If $\angle CBD = 30°$, prove that CP trisects $\angle ACB$. **[13]**

6. A person starts from the origin $O(0,0)$ in the X-Y plane. He takes steps of **one unit** along the X-axis (positive as well as negative direction)or the Y-axis (positive as well as negative direction). Traveling in this manner, find the total number of ways he can reach $A(4,3)$ by using exactly 11 steps? **[15]**

7. Find all the real numbers x, y, z such that

$$\frac{1}{xy} = \frac{x}{z} + 1, \frac{1}{yz} = \frac{y}{x} + 1, \frac{1}{zx} = \frac{z}{y} + 1.$$ [**15**]

8. Find all the positive integers (x, y, z) such that $xyz = 5(x + y + z)$. **[15]**

♣♣♣