

21st Indian National Mathematical Olympiad, 2006

Time: 4 hours

February 05, 2006

Calculators / Mathematical tables / Slide rule are not allowed.

1. In a nonequilateral triangle ABC , the sides a, b, c form an arithmetic progression. Let I and O denote the incentre and circumcentre of the triangle respectively.
 - (i) Prove that IO is perpendicular to BI .
 - (ii) Suppose BI extended meets AC in K , and D, E are the midpoints of BC, BA respectively. Prove that I is the circumcentre of triangle DKE .

2. Prove that for every positive integer n there exists a **unique** ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a + b - 1)(a + b - 2) + a.$$

3. Let X denote the set of all triples (a, b, c) of integers. Define a function $f : X \rightarrow X$ by

$$f(a, b, c) = (a + b + c, ab + bc + ca, abc).$$

Find all triples (a, b, c) in X such that $f(f(a, b, c)) = (a, b, c)$.

4. Some 46 squares are randomly chosen from a 9×9 chess board and are coloured red. Show that there exists a 2×2 block of 4 squares of which at least three are coloured red.

5. In a cyclic quadrilateral $ABCD$, $AB = a, BC = b, CD = c$, $\angle ABC = 120^\circ$, and $\angle ABD = 30^\circ$. Prove that

$$(i) c \geq a + b; \quad (ii) |\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}.$$

6. (a) Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that $n < l^2 < \left(1 + \frac{1}{2005}\right)n$.
(b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l , such that $n < l^2 < \left(1 + \frac{1}{2005}\right)n$.

