

# Differential Equations - Problems

Hemant Bhate

- Express the following dynamical systems as coupled first order differential equations if necessary and hence find their equilibrium points:

a) Linear harmonic oscillator

$$\frac{d^2x}{dt^2} + w^2x = 0$$

b) Lorentz equation (model system for atmospheric convection)

$$dx/dt = \sigma(y - x)$$

$$dy/dt = rx - y - xz$$

$$dz/dt = xy - bz$$

- Convert the given equation of Duffing Oscillator into differential equations of First Order so that the system is

(a) Non Autonomous (explicit time dependent systems)

(b) Autonomous (Do not explicitly depend on time)

$$\frac{d^2x}{dt^2} + \frac{\alpha dx}{dt} + w^2x + \beta x^3 = f(t) \cos wt$$

- Find the general solutions and draw the phase portrait for the following linear systems:

a)  $dx/dt = x; dy/dt = y$     b)  $dx/dt = x; dy/dt = 2y$

c)  $dx/dt = -y; dy/dt = x$     d)  $dx/dt = -x + y; dy/dt = -y$

e)  $dx/dt = x; dy/dt = y; dz/dt = z$

f)  $dx/dt = -x; dy/dt = -y; dz/dt = z$

g)  $dx/dt = -y; dy/dt = x; dz/dt = -z$

- Find the general solution of the linear system

$$dx/dt = x$$

$$dy/dt = ay \quad \text{where } a \text{ is constant.}$$

Sketch the phase portraits for  $a = -1$ ,  $a = 0$  and  $a = 1$  and notice that the qualitative structure of the phase portrait is the same for all  $a < 0$  as well as for all  $a > 0$ , but it changes at the parameter value  $a = 0$ . This is called a bifurcation value.

- Consider a two dimensional dynamical system as below. Find the eigenvalues of the system and hence find the form of solution assuming  $X^* = (x_0, y_0)$  to be the equilibrium point of the system.

$$dx/dt = P(x, y)$$

$$dy/dt = Q(x, y). \quad \text{where } P, Q \text{ are smooth functions.}$$

- Sketch the trajectories of the system  $X'(t) = AX(t)$ ,  $X(t) = (x_1(t), x_2(t))^T$ , for the following:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$$

7. Draw phase portraits for the following equation:  
Damped Cubic Anharmonic Oscillator:

$$\frac{d^2x}{dt^2} + \frac{ddx}{dt} - ax + bx^3 = 0, \quad a, b, d \text{ positive constants.}$$

Discuss the cases when (i)  $d^2 = 8a$  (ii)  $d^2 < 8a$  (iii)  $d^2 > 8a$

8. Undamped Pendulum equation : Sketch the phase portrait of the equation:

$$\frac{d^2x}{dt^2} + \sin x = 0$$

9. Sketch the phase portrait of the system :

$$dx_1/dt = -x_1 - 3x_2$$

$$dx_2/dt = 2x_2$$

10. Solve the system  $dx(t)/dt = Ax(t)$  for the following

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

11. Solve the linear system  $dX(t)/dt = AX(t)$  and determine the stable and the unstable subspaces for the linear system and sketch the phase portrait.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

12. If  $P$  and  $T$  are linear transformations on  $\mathbb{R}^n$  and  $S = PTP^{-1}$  then  $e^S = Pe^T P^{-1}$ .

Hence, if  $PAP^{-1} = \text{diag}[\lambda_i]$  then  $e^{At} = P \text{diag}[e^{\lambda_i t}] P^{-1}$ .

13. If  $S$  and  $T$  are linear transformations on  $\mathbb{R}^n$  which commute, then  $e^{S+T} = e^S e^T$ . Hence,  $(e^T)^{-1} = e^{-T}$

$$\text{Further, if } A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$\text{then } e^A = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$$

14. Compute the exponentials of the following matrices:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

15. Solve the system :

$$dx_1/dt = x_1$$

$$dx_2/dt = x_1 + 2x_2$$

$$dx_3/dt = x_1 - x_3$$

16. Show that the only solution to the system,

$$dx_1/dt = x_1$$

$$dx_2/dt = x_1 + x_2$$

$$x_1(0) = a, x_2(0) = b$$

is given by

$$x_1(t) = ae^t$$

$$x_2(t) = e^t(b + at)$$

17. Solve the initial value problems:

(a)  $dx/dt = -x$

$$dy/dt = x + 2y$$

$$x(0) = 0, y(0) = 3$$

(b)  $dx/dt = 2x + y$

$$dy/dt = x + y$$

$$x(1) = 1, y(1) = 1$$

18. Find a  $2 \times 2$  matrix  $A$  such that one solution to  $dx(t)/dt = Ax(t)$  is  $x(t) = (e^{2t} - e^{-t}, e^{2t} + 2e^{-t})$ .

19. Sketch the graphs of the three solutions of  $x'' - 3x' + 2x = 0$  for the initial conditions  $x(0) = 1, x'(0) = -1, 0, 1$ .

20. Consider the system  $x' = y, y' = -b^2x, b > 0, (x(0), y(0)) = (x_0, y_0)$ . Find the solution to the system and show that the solution curve  $(x(t), y(t))$  goes round and round an ellipse.

21. Find an example of two operators  $A$  and  $B$  on  $\mathbb{R}^2$ , such that  $e^{A+B} \neq e^B e^A$ .

22. If  $AB = BA$ , then  $e^A e^B = e^B e^A$  and  $e^A B = B e^A$ .

23. Show that there is no real  $2 \times 2$  matrix  $S$  such that  $e^S = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$

24. Solve the system using  $S + N$  Decomposition:

$$dx/dt = -x + y - 2z$$

$$dy/dt = -y + 4z$$

$$dz/dt = z.$$

25. For each of the operators  $T$ , find basis for the generalised eigenspaces; give the matrices (for standard basis) of the semisimple and nilpotent parts of  $T$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 0 & 4 \\ 0 & 1 & -2 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

26. The semisimple and nilpotent parts of  $T$  commute with  $A$  if  $T$  commutes with  $A$
27. What can be said about  $AB$  and  $A + B$  if  $AB = BA$  and
- $A$  and  $B$  are nilpotent.
  - $A$  and  $B$  are semisimple.
  - $A$  is nilpotent and  $B$  is semisimple
28. A rotation around a line in  $\mathbb{R}^3$  and reflection in a plane in  $\mathbb{R}^3$  are semisimple operators.
29. If  $T^2 = T$  then  $T$  is diagonalisable.
30. Each of the following properties defines a set of real  $n \times n$  matrices. Find out which sets are dense, and which are open in  $L(\mathbb{R}^n)$  of all linear operators in  $\mathbb{R}^n$ .
- determinant  $\neq 0$
  - trace is Rational
  - $3 \leq \text{determinant} < 4$
31. For each of the following functions find a Lipschitz constant on the region indicated if at all it exists.
- $f(x) = |x|, x \in \mathbb{R}$
  - $f(x) = x^{1/3}, |x| \leq 1$ .
32. Consider the differential equation  $x' = x^{2/3}$
- There are infinitely many solutions satisfying  $x(0) = 0$  on every interval  $[0, \beta]$
  - For what values of  $\alpha$  are there infinitely many solutions on  $[0, \alpha]$  satisfying  $x(0) = -1$ ?
33. Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Lipschitz and  $g : \mathbb{R} \rightarrow \mathbb{R}$  continuous. Show that the system

$$\begin{aligned} x' &= g(x) \\ y' &= f(x)y, \end{aligned}$$

has at most one solution on any interval, for a given initial value. (Hint: Use Gronwall's inequality).