

Pune, June 2006

Topics: Affine varieties, Hilbert's Nullstellensatz, affine and projective plane curves, multiplicities, singularities, Bezout's theorem, rational curves, cubic curves, ...

Additional references:

- William Fulton, Algebraic Curves.
- C. Musili, Algebraic Geometry.

Notation:

- k is an algebraically closed field of characteristic zero.
- $a, b, c \in k$.
- X, Y, Z, t are indeterminates.
- f, g, h are polynomials in $k[X, Y]$ (of positive degrees if the context so requires).
- F, G, H are homogeneous polynomials in $k[X, Y, Z]$ of positive degrees.
- R is a ring (commutative with 1).

Exercises:

- (1) If $f(a, b) = 0$ for all $a, b \in k$ then $f = 0$.
- (2) $f(a, b) = 0$ if and only if f belongs to the ideal $(X - a, Y - b)$.
- (3) $k[X, Y]_{(X-a, Y-b)}/(f(X, Y))$ is isomorphic to $k[X, Y]_{(X, Y)}/(f(X + a, Y + b))$ as a k -algebra, i.e. as a ring as well as a k -vector space. This shows that the local ring of the curve $f = 0$ at (a, b) is isomorphic to the local ring of the curve $f(X + a, Y + b) = 0$ at $(0, 0)$.
- (4) Let d equal the order of f as a formal power series in $X - a, Y - b$, and let e be least nonnegative integer such that a partial derivative of f of order e does not vanish at (a, b) . Then $d = e$. This number is $\text{mult}_{(a,b)}(f)$.
- (5) Let A be the local ring of the curve $f = 0$ at $P = (a, b)$, and let \mathfrak{m} be the maximal ideal of A . Then $\dim_k \mathfrak{m}^n / \mathfrak{m}^{n+1} = \text{mult}_P(f)$ for $n \gg 0$; more precisely, for $n \geq \text{mult}_P(f) - 1$.
- (6) All irreducible factors of F are homogeneous.
- (7) $F(X, Y, 0) = \prod_i (a_i X + b_i Y)$ with $a_i, b_i \in k$.

- (8) (With respect to Z) the **dehomogenization** of F is defined to be $F_* = F(X, Y, 1)$, and the **homogenization** of $f \neq 0$ is defined to be $f^* = f_d + Zf_{d-1} + \cdots + Z^{d-i}f_i + \cdots + Z^{d-1}f_1 + Z^d f_0$, where $f = f_d + f_{d-1} + \cdots + f_1 + f_0$ with $d = \deg(f)$ and each f_i homogeneous of degree i . Prove the following:
- $(FG)_* = F_*G_*$ and $(fg)^* = f^*g^*$.
 - $(f^*)_* = f$ and $Z^r(F_*)^* = F$ for some $r \geq 0$.
 - Suppose Z does not divide F . Then the irreducible factors of F_* are precisely the dehomogenizations of the irreducible factors of F .
 - $\gcd(F, G) = 1 \Rightarrow \gcd(F_*, G_*) = 1$.
 - $\gcd(f, g) = 1 \Rightarrow \gcd(f^*, g^*) = 1$.
 - if $f = g + h$ then $Z^p f^* = Z^q g^* + Z^r h^*$ for some $p, q, r \geq 0$.
- (9) Ideals I and J of R are said to be **comaximal** if $I + J = R$.
- If I and J are comaximal then $IJ = I \cap J$.
 - Let I_1, \dots, I_r be pairwise comaximal ideals of R , and let $J = \bigcap_i I_i$. Then the map

$$R/J \rightarrow R/I_1 \times \cdots \times R/I_r$$

given by $a + J \mapsto (a + I_1, \dots, a + I_r)$ is an isomorphism of rings (Chinese Remainder Theorem).

- (10) If \mathfrak{m} is a maximal ideal of R and I is an \mathfrak{m} -primary ideal of R then $R_{\mathfrak{m}}/IR_{\mathfrak{m}} = R/I$.
- (11) If R is artinian with $\text{Max}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ then R is isomorphic to $R_{\mathfrak{m}_1} \times \cdots \times R_{\mathfrak{m}_r}$.
[Hint: Use the previous two exercises.]
- (12) (a) \mathbb{A}_k^2 is not a union of finitely many affine curves.
(b) \mathbb{P}_k^2 is not a union of finitely many projective curves.
- (13) Given two distinct points P, Q in \mathbb{P}_k^2 there is a unique line in \mathbb{P}_k^2 passing through P and Q .
- (14) Prove the following assertions without using Bezout's Theorem:
- Two distinct lines in \mathbb{P}_k^2 meet in a unique point in \mathbb{P}_k^2 .
 - If F is not divisible by Z then the projective curves $F = 0$ and $Z = 0$ meet in only finitely many points in \mathbb{P}_k^2 .
- (15) If $0 \rightarrow V_0 \rightarrow V_1 \rightarrow \cdots \rightarrow V_n \rightarrow 0$ is an exact sequence of linear maps of finite-dimensional k -vector spaces then $\sum_{i=0}^n (-1)^i \dim_k(V_i) = 0$.
- (16) Let S_r be the k -vector space consisting of (zero and) all homogeneous polynomials in $k[X, Y, Z]$ of degree r . Prove the following:
- $\dim_k S_r = \binom{r+2}{2}$.

(b) If $\gcd(F, G) = 1$, $m = \deg(F)$, $n = \deg(G)$ and $r \geq m + n$ then the sequence

$$0 \rightarrow S_{r-m-n} \xrightarrow{\varphi} S_{r-m} \oplus S_{r-n} \xrightarrow{\psi} S_r \xrightarrow{\eta} S_r / (S_r \cap (F, G)) \rightarrow 0,$$

where $\varphi(H) = (HG, -HF)$, $\psi(K, L) = KF + LG$ and η is the natural map, is exact and

$$\dim_k S_r / (S_r \cap (F, G)) = mn.$$

- (17) An irreducible projective plane curve of degree one or two is nonsingular.
- (18) An irreducible projective plane curve of degree three can have at most one singularity, and if such a singularity exists then it is a double point.
- (19) (a) $\text{mult}_P(F) \leq \deg(F)$ for every point P on the projective curve $F = 0$.
 (b) If equality holds at some point P then the curve consists of $\deg(F)$ lines (not necessarily distinct).
- (20) If F is irreducible of degree at least two then $\text{mult}_P(F) \leq \deg(F) - 1$ for every point P on the projective curve $F = 0$.
- (21) A line $L = 0$ in the projective plane is tangent to the curve $F = 0$ at a point P if and only if $I(P, L \cap F) > \text{mult}_P(F)$.
- (22) Determine all singularities (including those at infinity) of the following affine plane curves. Also determine the multiplicity and all tangents lines at each singular point.
- (a) $X^2 + Y^2 = 1$.
 (b) $Y^2 = 1 + X^3$.
 (c) $Y^2 = X^2 + X^3$.
 (d) $Y^2 = X^3$.
 (e) $Y^2 = X^5$.
 (f) $Y^m = X^n$, $\gcd(m, n) = 1$.
 (g) $(Y^2 + X^2)^3 - X^2Y^2 = 0$.
- (23) For each of the following pair of affine or projective plane curves determine their points of intersection, compute intersection multiplicity at each point and verify Bezout's Theorem. If the curves are affine, do the exercise for their projective closures.
- (a) $Y^2Z = Z^3 + X^3$ and $Z = 0$.
 (b) $Y^3 = X^5$ and $Y = 0$.
 (c) $Y^3 = X^5$ and $X = 0$.
 (d) $Y^3 = X^5$ and $Y + X = 0$.
 (e) $Y = X^3$ and $Y = X^4$.
- (24) If $\beta(t)^2 = 1 + \alpha(t)^3$ with $\alpha(t), \beta(t) \in k(t)$ then $\alpha(t), \beta(t) \in k$. This proves that the curve $Y^2 = 1 + X^3$ is not rational over k .