BHASKARACHARYA PRATISHTHANA

Tutorial 3 for ATML 2007

CONFORMALITY, LINEAR FRACTIONAL TRANSFORMATIONS

Q.1 Prove that conformality (along with continuity of the partial derivatives implies holomorphicity). Prove, in fact, that preservation of angles between curves alone is sufficient, while constancy of the scaling factor in all directions is almost sufficient in that, a function with this property is either holomorphic or conjugate holomorphic.

Q.2 Determine the angle through which the tangents to all curves passing though the point $2+i$ are rotated under the transformation $w = z^2$.

Q.3 Determine which of the following transformations preserve the oriented angles between smooth curves passing through the origin.

(A) $u = x, v = -y$ (B) $u = 3x, v = 2y$

(C) $u = -5y, v = 5x$ (D) $u = x^2 - y^2, v = 2xy$

Q.4 Under the transformation $w = z^2$, determine (i) the image of a sector of a circle of radius r centred at O, (ii) the image of the semi-infinite strip $\{x + iy : 0 \leq x \leq k, y \geq 0\}$ and (iii) the inverse image of the rectangle $\{u + iv : a \leq u \leq b, c \leq v \leq d\}$. Sketch.

Q.5 Prove that the transformation $w = 1/z$ maps a straight line onto a straight line or a onto circle depending upon whether the line passes through O or not. What about the image of a circle? Find the images of the discs of unit radii centred at (i) 0, (ii) 1 and (iii) $1+i$.

Q.6 Prove that every linear, fractional transformation (L.F.T.) is a composite of a translation, a rotation, a dilation (or a contraction) and an inversion.

Q.7 Let z_1, z_2, z_3, z_4 be four distinct, extended complex numbers. Define their **cross ratio** to be $(z_1, z_2, z_3, z_4) = \dfrac{z_1 - z_3}{z_1 - z_4}\dfrac{z_2 - z_4}{z_2 - z_3}$. Modify this definition suitably if any of the points is ∞. Note that the order of the points matters. Prove that the cross ratio is preserved under linear fractional transformations.

Q.8 For every complex number w other than $1, 0$, and ∞, prove that $w = (w, 1, 0, \infty)$. Hence show that given any three distinct complex numbers z_2, z_3, z_4, the unique L.F.T. which takes them to $1, 0$ and ∞ respectively is given by $T(z) = (z_1, z_2, z_3, z_4)$.

Q.9 Prove that every L.F.T. maps a circle onto a circle, where a 'circle' means either a straight line or (an ordinary) circle. [*Hint:* Use Problems 5 and 6 above.]

Q.10 Prove that four distinct complex numbers z_1, z_2, z_3, z_4 lie on a 'circle' if and only if the cross ratio (z_1, z_2, z_3, z_4) is a real number.

Q.11 Let C be a 'circle'. Two points z and z^* are said to be **symmetric** w.r.t. C if for every three distinct points z_2, z_3, z_4 on C, we have $(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$. Prove that symmetry is preserved under L.F.T.'s. Which points are symmetric to themselves ? (It turns out that if the condition for symmetry is satisfied for one choice of z_2, z_3, z_4, then it is satisfied for all choices. This will follow from the next question.)

12. In the last question, prove that if C is a straight line then z and z^* are symmetric w.r.t. C if and only if they are reflections of each other into C while if C is an (ordinary) circle with centre M and radius r and P, Q are points represented by z, z^* respectively then z, z^* are symmetric w.r.t. C if and only if P, Q lie on the same ray from M and $MP.MQ = r^2$. (This gives a geometric interpretation of symmetry and also shows that symmetry is independent of the three points z_2, z_3, z_4 on the 'circle'.)

Q.13 Prove that every L.F.T. which maps the (open) unit disc onto itself is of the form $T(z) = c\dfrac{z-a}{1-\overline{a}z}$ for some complex numbers a, c with $|a| < 1$ and $|c| = 1$.

Q.14 Map the region between the circles $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ conformally onto an infinite strip and then onto a half plane.

Q.15 Map the region in the first quadrant bounded by the coordinate axes and the hyperbola $y = 1/x$ conformally onto the upper half plane.

Q.16 Besides the Riemann sphere, there is another interpretation of the extended complex plane $\mathbb{C}^*$ called the **complex projective line**. It makes it easier to see what is really 'linear' in an L.F.T Consider an ordered pair (z_1, z_2) of (ordinary) complex numbers z_1, z_2 at least one of which is non-zero. If $z_2 \neq 0$, we associate the complex number z_1/z_2 to this pair. Otherwise we associate ∞. Note that the same complex number may be associated to many different pairs. Now suppose an extended complex number z corresponds to the pair (z_1, z_2). Then for any complex numbers a, b, c, d with $ad \neq bc$, we have $\dfrac{az+b}{cz+d} = \dfrac{az_1 + bz_2}{cz_1 + dz_2} = \dfrac{w_1}{w_2}$ where $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$. This is a well-defined, non-singular linear transformation of the complex two dimensional vector space into itself.

Q.17 Prove that the transformation $w = \sin z$ maps the semi-infinite strip $-\pi/2 \leq x \leq \pi/2$ bijectively onto the upper half plane $v \geq 0$. Identify the points where this transformation is conformal. Show that the images of the horizontal segments lie along confocal ellipses. Similarly study the transformations $w = \cos z$ and $w = \sinh z$.

Q.18 Obtain transformations that are bijective and conformal (except possibly at the boundary points) which map the upper half plane to (i) the unit disc, (ii) an infinite strip (iii) a semi-infinite strip, (iv) an infinite sector of a given angular width and (v) the region $\{(u,v) : u > 0, v > 0, uv \geq 1\}$.