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Tutorial 2 for ATML 2007

SEQUENCES, SERIES AND POWER SERIES

Q.1 Let $\{z_n\}_{n\geq 0}$ be a sequence of complex numbers, where for each $n \geq 0$, $z_n = x_n + iy_n$. (Often, we consider sequences defined only for $n \geq 1$. But in the case of series, especially the power series, the 0-th term is generally present. As far as convergence is concerned, it makes little difference whether we start with $n = 0$ or $n = 1$. The limit of a sequence is also independent of it. The sum of a (convergent) series does, of course, change if we drop the 0-th term.) Prove that the sequence $\{z_n\}_{n\geq 0}$ is convergent, a Cauchy sequence, bounded if and only if both the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ have the corresponding properties as sequences of real numbers. Show also that in the case of convergence, the real and the imaginary parts of the limit are precisely the limits of $\{x_n\}$ and $\{y_n\}$ respectively. Can there be an analogue of monotonically increasing/decreasing sequences for complex sequences?

Q.2 From the result in Q.1, elementary properties of sequences of complex numbers can be established either directly or by using the corresponding properties of sequences of real numbers, except those which involve monotonicity. In particular, prove the following properties using both the approaches:

(i) Every convergent sequence is bounded.

(ii) Every convergent sequence is a Cauchy sequence and conversely

(iii) Limits of complex sequences are compatible with addition, subtraction, multiplication and division (i.e. limit of the sum of two sequences is the sum of their limits etc.).

(iv) If $z_n \longrightarrow L$ as $n \longrightarrow \infty$, then $|z_n| \longrightarrow |L|$ as $n \longrightarrow \infty$. Prove that the converse is false in general, but true if $L = 0$.

(v) Let $w_n = \dfrac{z_0 + z_2 + \ldots + z_n}{n+1}$ for $n \geq 0$ and L be a complex number. Then $\{w_n\}$ converges to L if $\{z_n\}$ does, but not necessarily conversely.

Q.3 As in the case of real sequences, the complex sequences give handy characterisations of certain topological concepts. In particular, prove that:

(i) A subset A of the complex plane is closed if and only if it is sequentially closed, i.e. it has the property that if $\{z_n\}$ is a sequence in A and $z_n \longrightarrow L$ as $n \longrightarrow \infty$, then $L \in A$.

(ii) A complex (or real) valued function $f(z)$ defined in a neighbourhood of a point c in the complex plane is continuous at c if and only if it is sequentially continuous at c.

(Actually, these characterisations also hold in any euclidean space, and, more generally, in what are called **metric spaces**.)

Q.4 A **subsequence** of a complex sequence $\{z_n\}_{n \geq 0}$ is defined exactly as that for real sequences. Prove that a if $\{z_n = x_n + iy_n\}$ has a convergent subsequence, so do the sequences $\{x_n\}$ and $\{y_n\}$, but that the converse is false. What is the catch?

Q.5 Prove that every bounded complex sequence has a convergent subsequence, using the corresponding property for real sequences. (This is the **Bolzano-Weierstrass** version of completeness of the complex number system. For the real number system there are many other versions of completeness. But those based on the order structure of real numbers have no direct analogues for complex numbers. In particular, there is no such thing as the least upper bound property. Nor is there an analogue of the theorem that every monotonic, bounded sequence is convergent. But the Bolzano-Weierstrass version often serves the purpose as the next problem shows.)

Q.6 Using the Bolzano-Weierstrass property, prove that:

(i) Every continuous function on a closed, bounded subset A of the complex plane is bounded. Further, if the function is real valued, then it attains its bounds, i.e. has both a maximum and a minimum.

(ii) Every nested sequence of non-empty, closed bounded subsets of the complex plane has a non-empty intersection. (This is a generaliation of the Cantor's theorem for the real line. The Bolzano-Weierstrass property and its consequences above hold true in all euclidean spaces, but not in arbitrary metric spaces.)

Q.7 A well-known application of the completeness of the real line is the **fixed point property** of a closed bounded interval, which says that if $f : [a, b] \longrightarrow [a, b]$ is continuous then it has a fixed point, i.e. a point $c \in [a, b]$ such that $f(c) = c$. Give two proofs of this property, one based on the Intermediate Value Property and another based on the Bolzano-Weierstrass theorem. Which proof is better? [*Hint:* For the first proof, consider the function $g(x) = f(x) - x$. For the second, assume f has no fixed point. Colour a point x of $[a, b]$ red or blue according as $f(x) > x$ or $f(x) < x$. Show that every subdivision of $[a, b]$ contains a subinterval whose end-points are of different colours. Choose finer and finer subdivisions to get a sequence $\{a_n\}$ of red points and a sequence $\{b_n\}$ of blue points, with $|a_n - b_n| \to 0$ as $n \to \infty$. Apply sequential continuity of f to get a contradiction.]

Q.8 The analogue of the result above in higher dimensions is known as the **Brouwer fixed point theorem** which is quite non-trivial to prove. However, for the plane, an elementary proof can be given to show that every triangle T, say ABC, has the fixed point property, by the following steps:

(i) For a point z in the triangle, let $d_1(z), d_2(z), d_3(z)$ denote the perpendicular distances of z from the sides BC, CA, AB respectively. Prove that each $d_k(z)$ is continuous and z is determined uniquely by any two of these numbers. Prove further that if z, w are two distinct points in T then at least one of the three inequalities $d_1(w) < d_1(z)$, $d_2(w) < d_2(z)$ and $d_3(w) < d_3(z)$ must hold.

(ii) Suppose $f : T \longrightarrow T$ is continuous and has no fixed points. Colour a point $z \in T$ red if $d_1(f(z)) < d_1(z)$, blue if $d_1(f(z)) \geq d_1(z)$ but $d_2(f(z)) < d_2(z)$ and finally, green if $d_1(f(z)) \geq d_1(z)$ and $d_2(f(z)) \geq d_2(z)$. Show that every point is uniquely coloured and the colours red, blue, green appear nowhere on the sides BC, CA, AB respectively. In particular, it follows that the vertices A, B, C are red, blue and green respectively.

(iii) Suppose T is divided into subtriangles by drawing lines parallel to the sides. Prove that at least one of these subtriangles is trichromatic, i.e. that its three vertices are of different colours. (This result, known as the **Sperner's lemma** is the crux of the argument. The proof is easier to see if expressed in terms of elementary graph theory. Construct a graph with one vertex at the centre of each of the subtriangular regions and one more vertex anywhere outside the triangle T. Join two vertices by an edge if and only if the common boundaries of the corresponding regions include a line segment with one endpoint red and the other blue. Because of the colour restictions, the vertex corresponding to the exterior of T is of an odd degree. Hence there has to be another vertex of an odd degree.)

(iv) For every $n \in \mathbb{N}$, show that T contains three points z_n, w_n, u_n of all different colours such that no two of them are at a distance more than $1/n$. Show further that suitable subsequences of the sequences $\{z_n\}, \{w_n\}$ and $\{u_n\}$ converge to a common point, say $z^* \in T$.

(v) Using sequential continuity of f at z^*, show that it cannot be assigned any of the three colours, a contradiction.

Q.9. (i) Determine the values of z for which the series $\sum_{n=0}^{\infty}\left(\frac{z}{1+z}\right)^n$ and the series $\sum_{n=0}^{\infty}\left(\frac{2z}{1+z}\right)^n$ converge.

(ii) Determine the radius of convergence of the power series :
(a) $\sum_{n=0}^{\infty} n^p z^n$ $(p > 0)$, (b) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, (c) $\sum_{n=0}^{\infty} n! z^n$, (d) $\sum_{n=0}^{\infty} z^{n!}$.

(iii) If the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ is R, what is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^{2n}$ and of $\sum_{n=0}^{\infty} a_n^2 z^n$?

(iv) If $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have radii of convergence R_1, R_2, show that the radius of convergence of $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ is $\min\{R_1, R_2\}$ (when $R_1 \neq R_2$) while that of $\sum_{n=0}^{\infty} a_n b_n z^n$ is at least $R_1 R_2$. Give an example where it is higher.

(v) If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, express $\sum_{n=0}^{\infty} n^2 a_n z^n$ in terms of $f(z)$ and its derivatives.

Q. 10. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f(0) \neq 0$, then prove that there exists a unique power series $f(z) = \sum_{n=0}^{\infty} b_n z^n$ whose Cauchy product with the first power series is the constant power series 1. One would naturally expect that this new power series has $\frac{1}{f(z)}$ as its sum function. But even if $\sum_{n=0}^{\infty} a_n z^n$ has a positive radius of convergence, it is not easy to show directly that the power series $\sum_{n=0}^{\infty} b_n z^n$ also has a positive radius of convergence. Put differently, even though the sum and the product of two analytic functions at the same point are easily seen to be analytic at that point, and the reciprocal, when it exists, is also analytic, this fact is not easy to establish directly. The corresponding assertion for holomorphicity is very easy to establish. It is customary to say, therefore, that the power series $f(z) = \sum_{n=0}^{\infty} b_n z^n$ is the **formal inverse** of the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ (i.e. that it has the form of a power series).

Q.11 Using power series, obtain a closed form expression for the n-th **Fibonacci number** F_n. These numbers are defined recursively by the relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial values $F_0 = 0$ and $F_1 = 1$. (Thus the first few Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, ...) [*Hint:* Let $F(z) = \sum_{n=0}^{\infty} F_n z^n$. Use the given relation to show that $F(z) = \frac{z}{1 - z - z^2}$. Expand back in as a power series in z. This and the next two exercises are typical applications of power series to combinatorics. The present one is an instance of solving what is called a **recurrence relation**.]

Q.12 A coin has probability p of showing a head when tossed $(0 < p < 1)$. Show that $\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}$ is the average or 'expected' number of tosses till three consecutive heads show. [Hint: Let p_n be the probability that three consecutive heads appear at the n-th toss, but not earlier. The desired answer then is the infinite sum $\sum_{n=0}^{\infty} np_n$. But there is no easy closed form expression for p_n. Nevertheless, the answer equals

$P'(1)$ where $P(z) = \sum_{n=0}^{\infty} p_n z^n$. To find a closed form expression for $P(z)$ put $q = 1 - p$ and show that the sequence $\{p_n\}$ satisfies the recurrence relation $p_n = qp_{n-1} + pqp_{n-2} + p^2qp_{n-3}$ for all $n \geq 4$ with the initial values $p_0 = p_1 = p_2 = 0$ and $p_3 = p^3$.]

Q.13 Let r be a fixed positive integer. Suppose we have an unlimited supply of r types of objects. For a positive integer n, let a_n be the number of ways to choose n objects (repetitions being allowed freely). Show that a_n is precisely the coefficient of z^n in the expansion of $(1 - z)^{-r}$. Hence find a closed formula for a_n.

Q.14 Prove that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges everywhere in the complex plane. The sum function is denoted by $\exp(z)$.

(i) Prove that $\exp(z)$ is its own derivative. In fact show that it is the only such function which satisfies the initial condition that its value at 0 is 1.

(ii) Prove that for any complex numbers a, b, $\exp(a+b) = \exp(a)\exp(b)$. (This is usually done using (i). But a direct proof based on the Cauchy product of two series and the binomial theorem is also possible.)

(iii) Define $\cos z$ and $\sin z$ by

$$\cos z = \frac{\exp(iz) + \exp(-iz)}{2} \tag{1}$$

$$\text{and} \quad \sin z = \frac{\exp(iz) - \exp(-iz)}{2i} \tag{2}$$

Prove that for every z,

$$\exp(iz) = \cos z + i \sin z \tag{3}$$

(This is called **Euler's formula**. Normally, it is used when x is real. Note that $\sin x, \cos x$ and $\exp(x)$ are all real when x is real.)

(iv) Prove that $\sin z$ is an odd function and $\cos z$ is an even function. Show further that they satisfy the identities :

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{4}$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{5}$$
$$\text{and} \quad \sin^2 z + \cos^2 z = 1 \tag{6}$$

for all a, b, z. Finally, prove that $\frac{d}{dz}(\sin z) = \cos z$ and $\frac{d}{dz}(\cos z) = -\sin z$ for all z.

(v) Using the properties in (iv), prove that for real x, the function $\cos x$ has a smallest positive zero.

(vi) Let π be twice the least positive zero of $\cos x$. Prove that both the sine and the cosine functions are periodic with period 2π while the function $\exp(z)$ is periodic with period $2\pi i$.

(vii) For every complex number $z = x + iy$ with $|z| = 1$, prove that there exists some real θ such that $\exp(i\theta) = z$. Moreover any two such θ's differ by a multiple of 2π. Deduce that every non-zero complex number z can be expressed as $|z|\exp(i\theta)$ where θ is multivalued, any two values differing by a multiple of 2π. Every such θ is called an **argument** of z. The set of all such values is denoted by $arg\ z$. By convention, some semi-open interval of length 2π is fixed and the unique value of θ lying in this interval is called the **principal argument** of z and denoted by $Arg\ z$. The most standard choice for such an interval is $(-\pi, \pi]$ although sometimes it is taken as $[0, 2\pi)$.

(Power series are important sources of holomorphic functions. They also provide 'clean' definitions of the trigonometric functions, i.e. definitions not based on any geometric visualisation. The elementary practice is to take the naive definitions of $\sin\theta$ and $\cos\theta$ and then define $\exp(x+iy)$ as $e^x(\cos y + i\sin y)$, where e^x is defined as the inverse function of the real natural logarithm (which itself, is defined as a certain definite integral). The power series definition of $\exp(z)$ turns this sequence around. Note that the number e figures nowhere in this approach. We can *define* it as $\exp(1)$ i.e. as $\sum_{n=0}^{\infty} \frac{1}{n!}$. The property in (ii) above can then be used repeatedly to show that for every positive integer n, $\exp(n)$ is indeed $e \times e \times e \times \ldots \times e$ (n times), which is the 'physical' definition of e^n. Once this is done it is reasonable to denote $\exp(z)$ by e^z for any exponent z.)

Q.15 (i) For every positive real number x, prove that there is a unique real number a such that $e^a = x$. This unique real number is denoted by $\ln x$ and is called the **natural logarithm** of x.

(ii) For every non-zero complex number z, prove that all complex numbers w for which $e^w = z$ are of the form $\ln |z| + iarg\ z$. Every such number is called a **logarithm** of z and denoted by $\log z$ or by $\ln z$. The **principal logarithm** of z is defined as $\ln |z| + iArg\ z$ and denoted by Ln z or by Log z. Properties like $\ln(ab) = \ln a + \ln b$ are to be interpreted as equalities of sets rather than of particular complex numbers, More precisely, it says that every member of the L.H.S. is a sum of some two terms one coming from each set on the R.H.S. and conversely every such sum is a member of the L.H.S.

(iii) Give an example of a sequence $\{z_n\}$ of complex numbers which converges to a (non-zero) complex number z but Ln z_n does not converge to Ln z. What does this signify?

(iv) Let a, b be any complex numbers with $a \neq 0$. Then a^b is defined to be $e^{b \ln a}$. Note that in general this is an infinite set. Even then the standard properties of powers such as $a^{b+c} = a^b a^c$ can be proved as equalities of sets from the corresponding properties of logarithms. Prove that a^b is single valued when the exponent

b is an integer. How many distinct values does it have when b is a rational number? In general, $e^{b \operatorname{Ln} a}$ is called the **principal value** of a^b. Determine the principal value of i^i.

(v) Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

(vi) Prove that $e^{i\pi}+1=0$. (This is easy. But the equation has a special significance because the five numbers it relates arose at different times and in very different contexts in the history of mathematics. It is said that this equation convinced Euler that God exists!)

Q.16 Let n be a fixed positive integer.

(i) Prove that every non-zero complex number z has exactly n distinct n-th roots. Where are they located in the Argand diagram? (When $z=1$, these roots are called the **n-th roots of unity**, 'unity' being an old name for the number 1.)

(ii) An n-th root of unity, say ξ is called a **primitive** n-th root of unity if n is the smallest positive integer such that $\xi^n = 1$. Prove that every n-th root of unity is some power of ξ. Note in particular that $e^{2\pi i/n}$ is a primitive root of unity. Prove further that $e^{2\pi ik/n}$ is a primitive root of unity if and only if k is relatively prime to n. (Those familiar with elementary group theory will recognise that the n-th roots of unity form a cyclic group of order n and that the primitive n-th roots of unity are precisely the generators of this group.)

Q.17 Let z be a complex number with $|z|=1$. Suppose there exist integers $k_1<k_2<k_3<k_4$ such that $z^{k_1}+z^{k_2}+z^{k_3}+z^{z_4}=0$. Prove that z is a complex root of unity. [*Hint:* Use Exercise 8 of Tutorial 1.]

Q.18 The complex numbers have interesting applications to proving certain combinatorial and trigonometric identities. Here is a sample.

Let $n > 1$ be a fixed positive integer. Prove that :

(i) $\left[1 - \binom{n}{3} + \binom{n}{4} - \binom{n}{6} + \cdots\right]^2 + \left[\binom{n}{1} - \binom{n}{3} + \binom{n}{5} + \cdots\right]^2 = 2^n$ (Note that the sums are only ostensibly infinite since the binomial coefficient $\binom{n}{k}$ vanishes whenever $k > n$.)

(ii) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(\frac{2n+1}{2})\theta}{2\sin\frac{\theta}{2}} (0 < \theta < 2\pi)$

(iii) $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{1}{2}\cos\frac{\theta}{2} - \frac{\cos(\frac{2n+1}{2})\theta}{2\sin\frac{\theta}{2}} (0 < \theta < 2\pi)$

(iv) $(1 - z_1)(1 - z_2)\cdots(1 - z_{n-1}) = n$ where $z_1, z_2, \ldots, z_{n-1}$ are the n^{th} roots of unity other than 1.

(v) $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^n - 1}$. (JEE favourite) [Hint: Use (iv).]

(vi) $\cos\dfrac{2\pi}{n} + 2\cos\dfrac{4\pi}{n} + \ldots + (n-1)\cos\dfrac{2(n-1)\pi}{n} = \sum_{k=1}^{n-1} k\cos\dfrac{2k\pi}{n} = -\dfrac{n}{2}$

(vii) $\sum_{k=1}^{n-1}(n-k)\cos\dfrac{2k\pi}{n} = -\dfrac{n}{2}$.