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Local Homeomorphisms and coverings :

1. Let  $X$  be a compact  $n$ -dimensional manifold. Show that there exists no local homeomorphism  $\varphi : X \longrightarrow \mathbb{R}^n$ .
2.  $X = (S^1 \times S^1) - \{pt\}$ 
  - (i.) Show that there exists no homeomorphism  $\varphi : X \longrightarrow \varphi(X) \subset \mathbb{R}^2$  where  $\varphi(X)$  is an open subset of  $\mathbb{R}^2$ .
  - (ii.) Show that there exists a local homeomorphism  $\psi : X \longrightarrow \mathbb{R}^2$ .
3. Show that there exists a surjective local homeomorphism  $f : \mathbb{R}^2 \longrightarrow S^2$ .
4. Show that there exists no covering  $p : \mathbb{R}^2 \longrightarrow S^2$ .
5.  $\Sigma_g$  = a compact orientable surface of genus  $g$ . Assume that  $\Sigma_g$  is so placed in  $\mathbb{R}^3$  so that it is invariant under the map  $\vec{x} \mapsto -\vec{x}$ . Let  $U_g = \Sigma_g / (\vec{x} \mapsto -\vec{x})$ . Show that the quotient map  $p : \Sigma_g \longrightarrow U_g$  is a covering.
6.  $Y$  a 2-dimensional compact, connected surface  $\neq S^2, \mathbb{P}^2$ . Show that  $\exists$  a covering  $p : \mathbb{R}^2 \longrightarrow Y$ .

Construction of Differentiable manifolds :

7. Let  $U$  be an open set in  $\mathbb{R}^n$ . Let  $f : U \longrightarrow \mathbb{R}$  be a smooth function.

*Definition:*  $c \in \mathbb{R}$  is said to be a *critical value* of  $f$  if for some  $x_0 \in U$ , s.t.  $f(x_0) = c$ , we have  $\nabla f|_{x_0} = 0$ . In this case  $x_0$  is called a *critical point* of  $f$ . If  $c$  is not a critical value then  $c$  is called *regular value* of  $f$ .

Show that if  $c$  is a regular value of  $f$ , then  $f^{-1}(c)$  has a canonical structure of an  $(n - 1)$ -dimensional differentiable manifold.

Give examples showing that a manifold obtained in this way may or may not be covered by a single co-ordinate chart.

8. Let  $U$  be an open set in  $\mathbb{R}^n$ . Let  $f : U \longrightarrow \mathbb{R}^k$ , be a smooth function. Let  $f = (f_1, f_2, \dots, f_k)$ .

*Definition:*  $c \in \mathbb{R}^k$  is said to be a *critical value* of  $f$  if for some  $x_0 \in U$ , s.t.  $f(x_0) = c$ , we have  $\text{rank of } Jf|_{x_0} < k$ . In this case  $x_0$  is called a *critical point* of  $f$ . If  $c$  is not a critical value then  $c$  is called a *regular value*.

Note:  $c$  is a regular value of  $f$ , if for all  $x \in U$  s.t.  $f(x) = c$ , we have  $\nabla f_i|_x, i = 1, 2, \dots, k$  are linearly independent.

Show that if  $c$  is a regular value then  $f^{-1}(c)$  has a canonical structure of an  $(n - k)$ -dimensional differentiable manifold.

*Definition :* A submanifold  $M^{n-k}$  of  $\mathbb{R}^n$  is said to be a *Differentiable Complete Intersection* if it is of the form  $f^{-1}(c)$  where  $f : U \longrightarrow \mathbb{R}^k$  is a smooth function, where  $U \subset \mathbb{R}^n$  is open and  $c$  is a regular value.

9. Let  $f \in \mathbb{R}[x_0, x_1, \dots, x_n]$  be a homogeneous polynomial of degree  $d$ . Let  $V(f) = \{x \in \mathbb{R}^{n+1} | f(x) = 0\}$ .

*Definition:* A *cone* in  $\mathbb{R}^{n+1}$  is a subset  $\tilde{C}$  of  $\mathbb{R}^{n+1}$  such that  $\tilde{C}$  contains a point  $\neq 0$ , and if  $x \in \tilde{C}$  then for all  $u \in \mathbb{R}$ ,  $ux \in \tilde{C}$ . (That is  $\tilde{C}$  is a non-empty union of lines in  $\mathbb{R}^{n+1}$  through the origin). To a cone  $\tilde{C}$  we can associate a subset  $C$  in  $\mathbb{P}^n(\mathbb{R})$ , the  $n$ -dimensional projective space.

Show that either  $V(f) = 0$ , or else it is a cone in  $\mathbb{R}^{n+1}$ .

In the latter case,  $V(f)$  is an example of what is called a *real algebraic variety*.

If  $x = (x_0, x_1, \dots, x_n)$  is a point  $\neq 0$  in  $\mathbb{R}^{n+1}$ , we denote by  $[x] = [x_0, x_1, \dots, x_n]$  its projection in  $\mathbb{P}^n(\mathbb{R})$ . By abuse of language, we call  $(x_0, x_1, \dots, x_n)$  the *homogeneous coordinates* of  $[x]$ . Note that these are *not* local coordinates in the sense we defined in the case of a manifold.

Notice that if  $f$  is a homogeneous polynomial of degree  $d$  in  $(x_0, x_1, \dots, x_n)$ , and  $d > 1$ , then 0 is critical value, for  $(0, 0, \dots, 0)$  is a critical point of  $f$ .

*Definition:* We say that 0 is a *projectively regular value* if  $V(f) \neq 0$ , and  $(0, 0, \dots, 0)$  in  $\mathbb{R}^{n+1}$  is the only critical point of  $f$ .

10. Suppose 0 is a *projectively regular value* of  $f$ , a homogeneous polynomial of degree  $d$  in  $(x_0, x_1, \dots, x_n)$ . Let  $S^n$  denote the unit sphere in  $\mathbb{R}^{n+1}$ , that is

$$S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}|^2 = 1\}$$

.

Show that  $SV(f) = S^n \cap V(f)$  is a compact  $(n - 1)$ -dimensional manifold.

(Hint: Exercise 8 and Euler's theorem).

11. Let  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^k$ ,  $f = (f_1, f_2, \dots, f_k)$ , be function, where  $f_i$  is a homogeneous polynomial of degree  $d_i, i = 1, 2, \dots, k$ . Let,

$$V(f) = \{x \in \mathbb{R}^{n+1} \mid f(x) = 0\}$$

Show that  $V(f) = 0$ , or else it is a cone in  $\mathbb{R}^{n+1}$ .

In the latter case,  $V(f)$  is again an example of what is called a *real algebraic variety*.

We extend the definition in exercise 8 by saying that 0 is a *projectively regular value* if  $V(f) \neq 0$ , and  $(0, 0, \dots, 0)$  is the only critical point of  $f$ .

Show that if  $c$  is a projectively regular value then  $f^{-1}(c)$  has a canonical structure of a compact  $(n - k)$ -dimensional differentiable manifold.

12. Let  $\tilde{M}^n$  be an  $n$ -dimensional differentiable manifold, and  $\Gamma$  a group of diffeomorphisms acting evenly,  $\Gamma \backslash \tilde{M}^n = M^n$  and  $p : \tilde{M}^n \longrightarrow M^n$  the canonical projection.
- Show that  $M^n$  admits a canonical structure of a differentiable manifold such that  $p$  is a smooth covering map.
13. Let  $X$  be a topological space, and  $\Gamma$  a group of homeomorphisms acting evenly. Let  $Y = \Gamma \backslash X$  and  $p : X \longrightarrow Y$  the canonical projection.
- Assume that  $Y$  has a structure of an  $n$ -dimensional differentiable manifold.
- Show that  $X$  admits a canonical structure of a differentiable manifold such that  $p$  is a smooth covering map.
14. Let  $p : X \longrightarrow Y$  be a covering map among topological spaces  $X$  and  $Y$ . We equip a subset of any topological space with the subspace topology. Let  $Z$  be a subset of  $Y$ , and  $W = p^{-1}(Z)$ . Show that  $p|_W : W \longrightarrow Z$  is a covering.
15. Let  $0$  be a projectively regular value of  $f$ , a homogeneous polynomial of degree  $d$  in  $(x_0, x_1, \dots, x_n)$ . Let  $SV(f) = S^n \cap V(f)$ , and  $PV(f) = p(SV(f))$  where  $p : S^n \longrightarrow \mathbb{P}^n(\mathbb{R})$  is the canonical projection.
- Show that  $PV(f)$  is a compact  $(n - 1)$ -dimensional differentiable manifold.
- (Hint: Show that the canonical projection of  $SV(f)$  on  $PV(f)$  is a double covering. Apply Exercises 9, 12, 13.)
16. Let  $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^k, f = (f_1, f_2, \dots, f_k)$ , be function, where  $f_i$  is a homogeneous polynomial of degree  $d_i, i = 1, 2, \dots, k$ . Suppose that  $0$  is a projectively regular value of  $f$ . Show that  $V(f)$  is a compact differentiable manifold.
17. Using the notion of “a tangent space at  $P$  to a differentiable manifold  $M^n$  as the best vector space approximation at  $P$ ”, but without calculation of any derivative, write the equation of a tangent line (resp. plane) to the following curves (resp surfaces).
- i) (the affine curve in  $\mathbb{R}^2$ )  $2x^2 + 3xy + y^3 - 15 = 0$  at  $(2, 1)$

- ii) (the projective curve in  $\mathbb{P}^2(\mathbb{R})$ )  $3x^3 + y^3 - 4z^2y = 0$  at  $(1, 1, 1)$
- iii) (the affine surface in  $\mathbb{R}^3$ )  $3x^3 + y^3 - 4z^2y = 0$  at  $(1, 1, 1)$
- iv) (the general conic in  $\mathbb{R}^2$ )  $ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0$ , at a point  $(x_0, y_0)$  (assuming that the conic represents a non-empty smooth curve).

18. Consider the  $n$ -dimensional real projective space  $\mathbb{P}^n(\mathbb{R})$  with homogeneous coordinates  $[x_0, x_1, \dots, x_n]$ . Let

$$A_i = \{[x_0, x_1, \dots, x_n] | x_i \neq 0\}$$

Let  $X_j = x_j/x_i$ . Show that  $(X_j), j \neq i$ , are coordinates on  $A_i$ , and  $A_i$  is homeomorphic to  $\mathbb{R}^n$ . Find the transition functions on  $A_i \cap A_j$  showing how  $\mathbb{P}^n(\mathbb{R})$  can be made into a differentiable manifold.

[[Note: This is a second proof showing the structure of a differentiable manifold on  $\mathbb{P}^n(\mathbb{R})$  from the first principles. Compare the first proof which used the idea of a covering space. ]]

19. Show the existence of a structure of a differentiable manifold on  $S^1$  from first principles.
20. Show the existence of a structure of a differentiable manifold on  $S^1 \times S^1$  from first principles.

[[The  $n$ -dimensional torus  $S^1 \times S^1 \times \dots \times S^1$  ( $n$  times) is one of the “standard spaces” in mathematics. The “angular coordinates”  $(\theta_1, \theta_2, \dots, \theta_n)$  can be used for most practical purposes, especially in the form of their exponentials or trigonometric functions, as is done in their use of Fourier series. However they are not co-ordinates in the sense of “co-ordinate charts” on differentiable manifolds.]]

21. Show the existence of a structure of a differentiable manifold on  $S^2$  from first principles.

[[ Hint: Think of stereographic projection.

Remark: In complex analysis of  $x + iy$  as a “co-ordinate” allowing it to have the value  $\infty$ . This may be called a “Möbius coordinate” on  $S^2$ . Note however that the  $(x, y)$  in “ $z = x + iy$ ” are not co-ordinates in the sense of “co-ordinate charts” on differentiable manifolds.]]

22. Let  $V$  be an  $(n+1)$ -dimensional vector space over  $\mathbb{R}$ . Let  $l : V \rightarrow \mathbb{R}$  be non-zero linear transformation. Let  $P(V)$  denote the projective space of 1-dimensional subspaces of  $V$ , and  $A_l = \{[x] \in P(V) \mid l(x) \neq 0\}$ . Setting the bijection of  $A_l$  with  $\{x \in V \mid l(x) = 1\}$ , show that  $A_l$  is diffeomorphic to  $\mathbb{R}^n$ .

Intuitive idea of Dimension:

23. By counting "degrees of freedom", justify on intuitive grounds the stated dimensions of the following spaces. Each of these spaces can be given a natural topology which turns it into manifolds of the stated dimensions. In fact, one can also turn each of these spaces into a differentiable manifold. One may think of equipping them with natural additional structures such as pseudo-Riemannian metrics, measures etc.

- The dimension of the space  $\mathcal{L}$  of lines in  $\mathbb{R}^2$  is 2.
- The dimension of the space  $\mathcal{L}^+$  of oriented lines in  $\mathbb{R}^2$  is 2.
- The dimension of the space  $\mathcal{C}^+$  of circles in  $\mathbb{R}^2$  is 3.
- The dimension of the space  $\mathcal{P}$  of parabolas in  $\mathbb{R}^2$  is 4.
- The dimension of the space  $\mathcal{E}$  of ellipses in  $\mathbb{R}^2$  is 5.
- The dimension of the space  $\mathcal{H}$  of hyperbolas in  $\mathbb{R}^2$  is 5.
- The dimension of the space  $\mathcal{T}$  of triangles in  $\mathbb{R}^2$  is 6.
- The dimension of the space  $\mathcal{T}_0$  of ordered triangles in  $\mathbb{R}^2$  is 6.
- The dimension of the space of congruence classes of triangles in  $\mathbb{R}^2$  is 3.
- The dimension of the space of similarity classes triangles in  $\mathbb{R}^2$  is 2.

*Remark :* The space of lines in  $\mathbb{R}^2$ ,  $\mathcal{L}$  can be identified with the *Möbius* strip.

A line in  $\mathbb{R}^2$  is given by the equation,  $ax + by + c = 0$  where  $a, b, c \in \mathbb{R}$  and  $(a, b) \neq (0, 0)$ . Here  $(a, b, c)$  and  $-(a, b, c)$  define the same line.

We can define the following map  $\varphi : \mathcal{L} \longrightarrow (S^1 \times \mathbb{R})/(\vec{x} \mapsto -\vec{x})$ ,

$$\varphi(a, b, c) = \left[ \left( \frac{(a, b)}{\|(a, b)\|}, c \right) \right]$$

$\varphi$  is one one and onto. We can put a topology on  $\mathcal{L}$  using the distance on  $\mathbb{R}^2$ , by saying points in  $\mathcal{L}$  are close if the corresponding lines in  $\mathbb{R}^2$

are close. With this topology  $\mathcal{L}$  is homeomorphic to the *Möbius* strip.

There is a two to one map from the space of oriented lines  $\mathcal{L}^+$  to  $\mathcal{L}$  which sends the oriented lines  $p + t\vec{v}$  and  $p + t(-\vec{v})$  to the same line. Considering the topologies on  $\mathcal{L}^+$  and  $\mathcal{L}$ , this is a 2-sheeted covering map. Now  $\mathcal{L}$  is homeomorphic to the *Möbius* strip  $\mathbb{M}$  which has  $S^1 \times \mathbb{R}$  as its unique connected double cover. Therefore  $\mathcal{L}^+$  is homeomorphic to  $S^1 \times \mathbb{R}$ .

24. Show that the space of circles in  $\mathbb{R}^2$ ,  $\mathcal{C}^+$  is homeomorphic to  $\mathbb{R}^2 \times \mathbb{R}_{>0}$ .
25. Show that the space of parabolas in  $\mathbb{R}^2$ ,  $\mathcal{P}$  is homeomorphic to  $\mathbb{R}^4 - \Delta$ , where  $\Delta = \{(X, Y) \in \mathbb{R}^4 \mid X, Y \in \mathbb{R}^2 \text{ and } X \neq Y\}$ .

Real Affine and Projective Quadrics:

26. Let  $A = [a_{ij}]$  be a symmetric  $(n+2) \times (n+2)$  matrix. Consider  $PS[A]$ , the subset of  $\mathbb{P}^{n+1}(\mathbb{R})$  given by  $\sum_{i,j} a_{ij}x_i x_j = 0$ . If  $\det A \neq 0$ , then  $PS[A]$  is called the non-singular projective quadric defined by  $A$ .

Show that  $PS[A]$  is non-empty iff neither  $A$  nor  $-A$  is positive definite.

27. Show that if  $PS[A]$  is non-empty then it is an  $n$ -dimensional compact differentiable manifold.

[[This happens iff  $A$  has signature  $(p+1, q+1)$ ,  $p+q = n$ , where  $p, q$  are non-negative integers. For convenience let  $(x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q)$  denote the homogeneous coordinates in  $\mathbb{P}^{n+1}(\mathbb{R})$ .]]

28. Suppose  $A, B$  are real symmetric  $(n+2) \times (n+2)$  matrices of signature  $(p+1, q+1)$ , where  $p, q$  are non-negative integers. Show that  $PS[A]$  is diffeomorphic to  $PS[B]$ .

[[We call  $PS[A]$  a projective quadric of signature  $(p, q)$ , and denote it by  $S^{p,q}$ . So, up to diffeomorphism, there is a unique  $n$ -dimensional projective quadric of signature  $(p, q)$ ,  $p+q = n$ .]]

29. Show that  $PS^{p,q}$  is diffeomorphic to  $PS^{q,p}$ . (So up to diffeomorphism we may assume  $p \geq q$ .) Show that if  $PS^{p,q}$  is diffeomorphic to  $PS^{p',q'}$  then  $(p, q) = (p', q')$ , or  $(p, q) = (q', p')$ . What is the number of  $n$ -dimensional projective quadrics, up to diffeomorphism?

30. Show that  $PS^{p,q}$  has a double cover which is diffeomorphic to  $S^p \times S^q$  where  $S^p$  denotes the unit sphere in  $\mathbb{R}^{p+1}$ . (Note  $S^0 = \{\pm 1\}$  consists of two points).

31. Consider the expression

$$(*) \sum a_{ij}x_i x_j + \sum a_i x_i + a_0,$$

where  $1 \leq i, j \leq n, a_{ij} = a_{ji}$ .

Show that by an affine change of coordinates  $(*)$  can be put in one of the following three forms.

- a)  $\sum \epsilon_i x_i^2$ , where  $1 \leq i \leq n$  and  $\epsilon_i = 1$ , or  $-1$ , or  $0$ .  
b)  $\sum \epsilon_i x_i^2 + x_{k+1}$ ,  $1 \leq i \leq k$  and  $\epsilon_i = 1$ , or  $-1$ ,

- c)  $\sum \epsilon_i x_i^2 + 1, 1 \leq i \leq k$  and  $\epsilon_i = 1$ , or  $-1$ , or  $0$ .
32. Let  $S(*)$  be the set of points in  $\mathbb{R}^n$  such that  $\sum a_{ij} x_i x_j + \sum a_i x_i + a_0 = 0$ . Determine when  $S(*)$  is empty.
33. For  $n = 1, 2$ , and  $3$  determine the homeomorphism types of  $S(*)$ .
34. Determine when  $S(*)$  is a differentiable submanifold of  $\mathbb{R}^n$ .
35. Write  $a_i = a_{i0} = a_{0i}$ , and  $a_0 = a_{00}$ . Consider the  $(n+1) \times (n+1)$  matrix  $A = [a_{ij}]$  where  $0 \leq i, j \leq n$ .  $S(*)$  is called a non-singular affine quadric if  $\det A \neq 0$ , and  $A$  is neither positive definite nor negative definite. Show that a non-singular affine quadric is an  $(n-1)$ -dimensional differentiable manifold
36. With the conventions in Exercise 30, we can associate to  $S(*)$  the non-singular projective quadric  $PS[A]$ . The signature of  $PS[A]$ , as defined in Exercise 6 is also called the signature of  $S(*)$ .
- Let
- $\mathcal{A}Q_n$  = the set of  $n$ -dimensional non-singular affine quadrics.
- $\mathcal{P}Q_n$  = the set of  $n$ -dimensional non-singular projective quadrics.
- We thus get a map of  $\mathcal{A}Q_n$  onto  $\mathcal{P}Q_n$ .
- For  $n = 1$  and  $2$ , describe the homeomorphism types of  $n$ -dimensional non-singular affine quadrics of a given signature.
37. Let  $\mathcal{A}(n+1)$  (resp.  $\mathcal{P}(n+1)$ ) denote the group of affine (resp projective transformations of  $(n+1)$ -dimensional vector space (projective space over  $\mathbb{R}$ ). Clearly  $\mathcal{A}(n+1)$  (resp.  $\mathcal{P}(n+1)$ ) acts on  $\mathcal{A}Q_n$  (resp.  $\mathcal{P}Q_n$ ). In each case describe the orbits, and show that there are only finitely many orbits.
38. Let  $V$  be an  $(n+1)$ -dimensional vector space over  $\mathbb{R}$ , equipped with a positive inner product. Let  $\mathcal{I}(n+1)$  (resp  $\mathcal{S}(n+1)$  ) be the group of isometries (resp similarities) of the associated affine space. For  $n=1$ , describe the orbits of  $\mathcal{I}(2)$  (resp.  $\mathcal{S}(2)$  ) on each orbit of  $\mathcal{A}(2)$  on  $\mathcal{A}Q_n$ .
39. Do exercise 37 for  $n = 2$ .

Asymptotes of a Plane Curve:

*Euclidean definition:* Let  $C$  be a subset of  $\mathbb{E}^2$ , the Euclidean plane. Let  $l$  be a line in a parametric form:  $t \mapsto P_t = \vec{a} + t\vec{v}$ ,  $-\infty < t < \infty$ . Distance between a point  $P \in \mathbb{E}^2$  and  $C$  is defined to be  $\inf_{Q \in C} d(P, Q)$ . The line  $l$  is said to be an asymptote to  $C$  if either  $\lim_{t \rightarrow \infty} d(P_t, C) = 0$ , or  $\lim_{t \rightarrow -\infty} d(P_t, C) = 0$ .

40. Let  $C$  be the curve given by the analytic expression " $xy = 1$ ". Guess all of its asymptotes, and prove your guess.
41. Show that " $l$  is an asymptote of  $C$ " is an affine notion. (You also need to interpret what "affine notion" means.)
42. Show that "to be an ellipse", "to be a hyperbola" and "to be a parabola" are affine notions.
43. Show that a hyperbola has two asymptotes, but neither ellipse, nor parabola has any asymptotes.
44. Interpret the meaning of asymptotes of a hyperbola, and their non-existence for an ellipse or parabola in projective terms.

Dynamics of Linear and Affine maps :

45. Consider  $f, g : \mathbb{R} \rightarrow \mathbb{R}, f(x) = ax, g(x) = ax + b$ . Describe the dynamics of  $f$  and  $g$ .  
(Hint: You need to consider cases according to the signs and magnitudes of  $a$  and  $b$ . Look for their dynamic significance.)
46. Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $\vec{x} \mapsto A\vec{x}$ .  
Describe the dynamics of  $f$ . In this description carefully define the notions of (i) stretch and bi-stretch, (ii) rotation, (iii) shear and (iv) inversion.
47. Let  $A = [a_{ij}], 1 \leq i, j \leq n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  given by  $\vec{x} \mapsto A\vec{x}$ . Using the Jordan-Chevalley canonical form, describe the dynamics of  $f$ .
48. Let  $A$  be as in exercise 46 and  $\vec{b} \in \mathbb{R}^n$ . Let  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by  $g(\vec{x}) = A\vec{x} + \vec{b}$
- (i) Show that if  $\det(A - I) \neq 0$ , then  $g$  has a fixed point.
  - (ii) Describe the dynamics of  $g$ .
- (Hint: In addition to the notions of stretch, rotation, shear, inversion you will also need the notion of translation.)
49. In the situation of exercise 46, suppose that the minimal polynomial of  $A$  factors into pairwise distinct irreducible factors. Describe the dynamics of  $f$ .
50. In the situation of exercise 47 suppose that the minimal polynomial of  $A$  factors into pairwise distinct irreducible factors. Describe the dynamics of  $g$ .
51. In the situation of exercise 46 suppose that  $A$  has no eigenvalue (real or complex) of absolute value 1. Show that  $\mathbb{R}^n$  splits into two subspaces:  $\mathbb{R}^n = V_s \oplus V_u$  such that for each  $v \in V_s$  (resp  $V_u$ )  $\lim_{n \rightarrow \infty} A^n v = 0$  (resp  $\lim_{n \rightarrow \infty} A^{-n} v = 0$ ).

*Remark:*  $V_s$  (resp  $V_u$ ) are known as stable (resp unstable) subspaces.

ODE's and Flows :

52. (i) Solve  $\frac{dx}{dt} = ax$  (1-variable case)  
(ii) Solve the (IVP)  
(\*)  $\frac{dx}{dt} = ax, x(0) = x_0$   
(iii) Write the flow  $\varphi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  corresponding to (\*).  
(iv) Verify directly  $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ .
53. Do i), ii), iii) and iv) of exercise 51 for  $\frac{dx}{dt} = ax + b$ .
54. Describe the flows generated by the following vector fields on  $\mathbb{R}$ .
- (i)  $X = \frac{\partial}{\partial x}$ .  
(ii)  $Y = x \frac{\partial}{\partial x}$ .  
(iii)  $Z = x^2 \frac{\partial}{\partial x}$ .

Which of these are complete? Draw pictures.

55. Describe the flow on  $\mathbb{R}$  generated by the vector field  $Y = \sin x \frac{\partial}{\partial x}$ . Is it complete? Draw pictures.
56. Can the vector field in exercise 55 be considered a vector field on  $S^1$ ? If so what can you say about the flow.
57. Let  $S = \{a_n | n \in \mathbb{Z}\}$  be real numbers such that  $\lim_{n \rightarrow \infty} a_n = \infty$  and  $\lim_{n \rightarrow -\infty} a_n = -\infty$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function whose zero is  $S$ . Let  $X = f(x) \frac{\partial}{\partial x}$ . Show that  $X$  is complete and describe its flow.
58. Let  $M^n$  be a differentiable manifold, and  $V$  a compact subset of  $M^n$ . Let  $X$  be a smooth vector field on  $M^n$ . Show that there is a "local flow".  
i.e.  $\exists \epsilon \geq 0$  and an open neighbourhood  $U$  of  $V$  and a correspondence

$$\varphi : (-\epsilon, \epsilon) \times U \rightarrow M^n$$

such that if  $\varphi(t, x) = \varphi_t(x)$  then

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$$

for all  $t, s \in (-\epsilon, \epsilon)$  and all  $x \in U$  such that  $\varphi_s(x) \in U$ .

Note:  $\varphi$  defines a “motion” of  $V$  in  $M^n$ .

59. *Recall:* Let  $M^n$  be a differentiable manifold and  $X$  be a smooth vector field on  $M^n$ . Given  $P \in M^n$   $\exists \epsilon_P > 0$  and a neighbourhood  $U_P$  of  $P$  and a correspondence

$$\varphi : (-\epsilon, \epsilon) \times U \rightarrow M^n$$

such that

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$$

for all  $t, s \in (-\epsilon_P, \epsilon_P)$  and all  $x \in U$  such that  $\varphi_s(x) \in U$ .

Show that , if  $\inf_{P \in M^n} \epsilon_P > 0$  then  $X$  is a complete vector field.

60. Let  $M^n$  be a compact differentiable manifold. Show that every smooth vector field on  $M^n$  is complete.
61. Let  $M^n$  be a differentiable manifold and  $X$  a vector field with compact support. Show that  $X$  is complete.
62. Show that the vector fields in exercise 53 are closed under the Lie bracket.

Note: These vector fields form a Lie algebra. It is the Lie subalgebra of the Lie algebra of all vector fields on  $\mathbb{R}$ .

63. Show that the vector fields  $\frac{\partial}{\partial x}$ ,  $\sin x \frac{\partial}{\partial x}$  and  $\cos x \frac{\partial}{\partial x}$  on  $\mathbb{R}$  (or  $S^1$ ) form a Lie algebra .
64. Describe the flows on  $\mathbb{R}^2$  generated by the vector fields

(i)  $X_1 = 2x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y}$ .

$$(ii) X_2 = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

$$(iii) X_3 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

$$(iv) X_4 = x \frac{\partial}{\partial y}$$

Are the vector fields complete?

Remark : Let  $M^n$  be a differentiable manifold. If  $X$  is a complete vector field on  $M^n$  then it generates a smooth  $\mathbb{R}$ -action on  $M^n$ . Conversely a smooth  $\mathbb{R}$ -action gives rise to a complete vector field. So to generate interesting examples we may as well start with interesting smooth  $\mathbb{R}$ -actions.

More generally let  $G$  be a connected Lie group acting smoothly on  $M^n$ , then smooth homomorphisms of  $\mathbb{R}$  into  $G$  generate smooth  $\mathbb{R}$ -actions on  $M^n$  and in turn complete vector fields.

The information about smooth homomorphisms of  $\mathbb{R}$  into  $G$  is just the information about  $G$  itself. This information will get reflected in *every* smooth  $G$ -action on *every* differentiable manifold  $M^n$ .

*Example 1* Let  $M^n = \mathbb{R}^n, G = GL_n(\mathbb{R})$ . The action by invertible linear transformations  $\vec{x} \mapsto A\vec{x}$ .

*Example 2* Let  $M^n = \mathbb{R}^n, G = Aff_n(\mathbb{R})$ , where,

$$Aff_n(\mathbb{R}) = \{f : \mathbb{R}^n \longrightarrow \mathbb{R}^n | f(\vec{x}) = A\vec{x} + \vec{b}, A \in GL_n(\mathbb{R}), \vec{b} \in \mathbb{R}^n\}.$$

*Example 3* Example 1 in turn induces the action of  $G$  on  $S^{n-1}$  considered as the “space of rays in  $\mathbb{R}^n$  through the origin”. It may be expressed as  $\vec{u} \in S^{n-1} \mapsto \frac{A\vec{u}}{\|A\vec{u}\|} \in S^{n-1}$ .

*Example 4* The action of  $G$  in Example 3 induces the action on  $\mathbb{P}^{n-1}(\mathbb{R})$ . In homogeneous coordinates it looks like  $[\vec{x}] \mapsto [A\vec{x}]$ .

Homomorphisms of  $\mathbb{R}$  into  $\mathbb{R}^n, S^1$  and  $S^1 \times S^1$  :

65. Show that a  $C^0$ -homomorphism of  $\mathbb{R}$  into  $\mathbb{R}^n$  is “standard”. (A part of this exercise is to define “standard”) Is the result true if we drop the hypothesis  $C^0$ ?  $\mathbb{R}$  has a canonical structure of a real analytic ( $C^\omega$ ) manifold. Show that a  $C^0$ -homomorphism is already  $C^\omega$ .
66. Consider  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ . Here  $2\pi\mathbb{Z}$  is a normal subgroup of  $\mathbb{R}$ . We have a canonical group homomorphism  $p : \mathbb{R} \rightarrow S^1$ .
- (i) Show that  $p$  is a covering map.
  - (ii) Show that  $S^1$  has a unique structure of a real analytic manifold such that  $p$  is real analytic.
  - (iii) Classify  $C^0$ -homomorphisms of  $\mathbb{R}$  into  $S^1$ . Show that they are all  $C^\omega$ .
67. Classify all  $C^0$  homomorphisms of  $S^1$  into  $S^1$ .
68. (Subgroups of  $\mathbb{R}$ )
- (i) Show that a discrete subgroup of  $\mathbb{R}$  is either  $\{0\}$  or isomorphic to  $\mathbb{Z}$ .
  - (ii) Construct an injective homomorphism  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{R}$ . Show that the image of such a homomorphism must be dense.
  - (iii) Construct injective homomorphisms of  $\mathbb{Z}^n$  into  $\mathbb{R}$  for  $n = 1, 2, 3, \dots$
69. Classify all  $C^0$ -homomorphisms of  $\mathbb{R}$  into  $S^1 \times S^1$ . Show that the image of  $\mathbb{R}$  under a  $C^0$ -homomorphism  $\varphi : \mathbb{R} \rightarrow S^1 \times S^1$  is either dense or it is a closed subgroup.
- (i) Determine when the image of  $\mathbb{R}$  under a  $C^0$ -homomorphism  $\varphi : \mathbb{R} \rightarrow S^1 \times S^1$  is dense in  $S^1 \times S^1$ . In this case,  $\varphi$  is a group theoretic isomorphism  $\mathbb{R}$  onto  $\varphi(\mathbb{R})$ , but not a homeomorphism of  $\mathbb{R}$  onto  $\varphi(\mathbb{R})$ , where  $\varphi(\mathbb{R})$  is given the induced topology.
  - (ii) Determine when the image of  $\mathbb{R}$  under a  $C^0$ -homomorphism  $\varphi : \mathbb{R} \rightarrow S^1 \times S^1$  is closed in  $S^1 \times S^1$ . In this case  $\ker\varphi \neq \{e\}$ .

Homomorphisms of  $\mathbb{R}$  into  $GL_n(\mathbb{R})$  :

70. Using the existence-uniqueness theorem on ODE, show that every  $C^2$ -homomorphism of  $\mathbb{R}$  into  $GL_n(\mathbb{R})$  is of the form  $t \mapsto e^{tA}$  for some  $A = (a_{ij}) \in M_n(\mathbb{R})$ .
71. Let  $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid \det A = 1\}$ . Consider a homomorphism  $\varphi : t \mapsto e^{tA}$  of  $\mathbb{R}$  into  $GL_n(\mathbb{R})$ . Show that  $\varphi$  lies in  $SL_n(\mathbb{R})$  iff  $\text{trace } A = 0$ .
72. Let  $O(n) = \{A \in GL_n(\mathbb{R}) \mid A^t A = I\}$ . Consider a homomorphism  $\varphi : t \mapsto e^{tA}$  of  $\mathbb{R}$  into  $GL_n(\mathbb{R})$ . Show that  $\varphi$  lies in  $O(n)$  iff  $A$  is skew-symmetric.

Homomorphisms of  $\mathbb{R}$  into  $Aff_n(\mathbb{R})$  :

*Remark* : Embedding of  $Aff_n(\mathbb{R})$  into  $GL_{n+1}(\mathbb{R})$  :

Consider the  $n$ -dimensional hyperplane  $x_{n+1} = 1$  in  $\mathbb{R}^{n+1}$ . Denote the map  $f : \vec{x} \mapsto A\vec{x} + \vec{b}$  by  $f_{A,\vec{b}}$ . Note,  $f_{A,\vec{b}} \circ f_{A',\vec{b}'} = f_{AA',A\vec{b}'+\vec{b}}$ . Consider

the element  $g_{A,\vec{b}} \in GL_{n+1}(\mathbb{R})$  defined by  $\begin{bmatrix} A & \vec{b} \\ 0 & 1 \end{bmatrix}$ .

73. Show that

- (i)  $\{g_{A,\vec{b}} | A \in GL_n(\mathbb{R}), \vec{b} \in \mathbb{R}^n\}$  is a subgroup of  $GL_{n+1}(\mathbb{R})$ . Call it  $\mathcal{A}$
- (ii)  $g_{A,\vec{b}}$  leaves the plane  $x_{n+1} = 1$  invariant.
- (iii)  $g \in GL_{n+1}(\mathbb{R})$  preserves  $x_{n+1} = 1$  iff  $g$  is of the form  $g_{A,\vec{b}}$  for some  $A \in GL_n(\mathbb{R}), \vec{b} \in \mathbb{R}^n$ .

74.  $f_{A,\vec{b}} \mapsto g_{A,\vec{b}}$  is an isomorphism of  $Aff_n(\mathbb{R})$  with  $\mathcal{A}$ .

75. Let  $A = (a_{ij})_{n \times n}, \vec{b} \in \mathbb{R}^n$ . Consider  $A_1 = \begin{bmatrix} A & \vec{b} \\ 0 & 0 \end{bmatrix}$

- (i) Show that  $e^{A_1} \in \mathcal{A}$ . In particular  $t \mapsto e^{tA_1}$  is a  $C^\omega$ -homomorphism of  $\mathbb{R}$  into  $\mathcal{A}$ .
- (ii) Show that  $e^{tA_1} = \begin{bmatrix} e^{tA} & \vec{b}(t) \\ 0 & 1 \end{bmatrix}$ , where
$$\vec{b}(t) = t\vec{b} + \frac{t^2}{2!}A\vec{b} + \dots + \frac{t^r}{r!}A^{r-1}\vec{b} + \dots$$

76. In exercise 74 let  $A = (a_{ij})_{n \times n}$  be an invertible matrix. Show that  $\vec{b}(t) = A^{-1}(e^{tA} - I)\vec{b}$

Flows generated by affine vector fields :

An Affine vector field on  $\mathbb{R}^n$  is a vector field of the form;

$$X = \sum_{i=1}^n \sum_{j=1}^n (a_{ij}x_j + b_i) \frac{\partial}{\partial x_i}$$

The corresponding affine system of ODE's is;

$$(*) \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}, \text{ where } A = (a_{ij}), \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

From a dynamic viewpoint, a pertinent question is whether this vector field has a zero, i.e. equivalently whether the corresponding flow has a fixed point. This motivates the following analysis.

*Case 1* :  $\exists \vec{x}_1 \in \mathbb{R}^n$  such that  $A\vec{x}_1 + \vec{b} = 0$ . (This is always the case when  $A$  is invertible, in fact if  $A$  is invertible,  $\vec{x}_1$  is unique).

Consider the affine change of coordinates :  $\vec{y} = \vec{x} - \vec{x}_1$ . Then,

$$\frac{d\vec{y}}{dt} = \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b} = A(\vec{x} - \vec{x}_1) + A\vec{x}_1 + \vec{b} = A\vec{y}$$

So in the  $\vec{y}$ -coordinate the flow is  $\varphi : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$  given by  $(t, \vec{y}) \mapsto e^{tA}\vec{y}$ . In particular the flow is complete.

Expression of the flow in terms of the  $\vec{x}$ -coordinate: Write  $V = \mathbb{R}^n$  as a space. Let  $P \in V$  be a point. Let the  $\vec{x}$ -coordinate of  $P$  be  $\vec{\xi}$  and the  $\vec{y}$ -coordinate be  $\vec{\eta}$ . We know  $\vec{\eta} = \vec{\xi} - \vec{x}_1$ .

Let  $\varphi_t(P) = Q$ . The  $\vec{y}$ -coordinate of  $Q$  is  $e^{tA}\vec{\eta}$ . So its  $\vec{x}$ -coordinate is  $e^{tA}\vec{\eta} - \vec{x}_1$ . As an expression in  $\vec{\xi}$  we get  $e^{tA}(\vec{\xi} - \vec{x}_1) + \vec{x}_1$ .

In other words the flow  $\varphi : \mathbb{R} \times V \longrightarrow V$  in  $\vec{x}$ -coordinates has the expression  $(t, \vec{x}) \mapsto e^{tA}(\vec{x} - \vec{x}_1) + \vec{x}_1$ .

Note: If  $A$  is invertible, the choice of  $\vec{x}_1$  is unique, i.e.  $\vec{x}_1 = A^{-1}\vec{b}$ . Then the expression of the flow is

$$(t, \vec{x}) \mapsto e^{tA}(\vec{x} - A^{-1}\vec{b}) + A^{-1}\vec{b} = e^{tA}\vec{x} + A^{-1}(e^{tA} - I)\vec{b}$$

This reconciles with the expression obtained in exercise 75.

*Case 2 : General case* Here we do not assume the existence of  $\vec{x}_1$  satisfying  $A\vec{x}_1 + \vec{b} = 0$ . The method given below works without any such assumption.

Let  $m_A(x)$  = the minimal polynomial of  $A$ . Write  $m_A(x) = x^r g(x)$  where  $g(0) = 0$ .

If  $r = 0$ , then  $A$  is invertible, this case is contained in the previous case.

Now,

$$(*) \quad \gcd(x^r, g(x)) = 1$$

Let,  $V_0 = \ker A^r$  and  $V_1 = \ker g(A)$ , then

$(\mathbb{R}^n =) V = V_0 \oplus V_1$ , this direct sum is  $A$ -invariant because of  $(*)$  and  $\vec{b} = \vec{b}_0 + \vec{b}_1$ , where  $\vec{b}_0 \in V_0$  and  $\vec{b}_1 \in V_1$  are uniquely determined.

Since  $A|_{V_1}$  is invertible, there exists a unique  $\vec{x}_1 \in V_1$  such that  $A\vec{x}_1 + \vec{b}_1 = 0$ . Consider the affine change of coordinates  $\vec{y} = \vec{x} - \vec{x}_1$ . Then;

$$\frac{d\vec{y}}{dt} = \frac{d\vec{x}}{dt} = A\vec{x} + \vec{b}_1 + \vec{b}_0 = A(\vec{x} - \vec{x}_1) + A\vec{x}_1 + \vec{b}_1 + \vec{b}_0 = A\vec{y} + \vec{b}_0,$$

where  $A^r \vec{b}_0 = \vec{0}$ .

In view of exercise 74, part (ii) we see that the flow in  $\vec{y}$ -coordinate is ;

$$\varphi : (t, \vec{y}) \mapsto e^{tA}\vec{y} + t\vec{b}_0 + \frac{t^2}{2!}A\vec{b}_0 + \cdots + \frac{t^r}{r!}A^{r-1}\vec{b}_0$$

In  $\vec{x}$ -coordinates the expression is

$$\varphi : (t, \vec{x}) \mapsto e^{tA}(\vec{x} - \vec{x}_1) + \vec{x}_1 + t\vec{b}_0 + \frac{t^2}{2!}A\vec{b}_0 + \cdots + \frac{t^r}{r!}A^{r-1}\vec{b}_0$$

77. Let  $\varphi_t : \mathbb{R}^n \longrightarrow \text{Aff}_n(\mathbb{R})$  be given by;  $\vec{x} \mapsto e^{tA}\vec{x} + \sum_{r=1}^{\infty} \frac{t^r}{r!}A^{r-1}\vec{b}$ . Show directly that for all  $t, s \in \mathbb{R}$ ,  $\varphi_{t+s} = \varphi_t \circ \varphi_s$

Orientability :

78. Show that  $GL_n(\mathbb{C})$  is connected.
79. Let  $M^n$  be a complex  $n$ -dimensional manifold. Show that as a real  $2n$ -dimensional manifold it is orientable.
80. Show that any complex vector bundle is orientable.
81. Show that any covering of an orientable differentiable manifold is orientable.
82. Let  $M^n$  be an  $n$ -dimensional orientable differentiable manifold. Suppose that  $G$  acts on  $M^n$  evenly and differentiably. Show that  $G \backslash M^n$  is orientable iff  $G$  is orientation-preserving.
83. Consider the map  $\sigma : x \mapsto -x$  on  $\mathbb{R}^{n+1}$ . Let  $S^n$  be the unit  $n$ -sphere in  $\mathbb{R}^{n+1}$  centered at 0. Show that
- i)  $\sigma$  is orientation-preserving iff  $n$  is odd.
  - ii) Let  $\tau$  resp  $\nu$  be the tangent (resp normal) bundle of  $S^n$ . Show that  $\nu$  is trivial and  $\sigma$  is orientation-preserving on  $\nu$ .  
[Hint:  $\sum x_i \partial / \partial x_i$  is a nowhere vanishing section of the normal bundle of  $S^n$ .]
  - iii)  $\mathbb{P}^n(\mathbb{R})$  is orientable iff  $n$  is odd.  
[Hint: Consider the action of  $\sigma$  on  $\tau$ . Use i) ii) and exercise 79.]
84. Let  $M^n$  be an  $n$ -dimensional non-orientable path-connected differentiable manifold then it admits a unique double cover which is orientable.
85. Let  $\mathcal{V}$  be a vector bundle of rank 1. Show that  $\mathcal{V}$  is orientable iff it is trivial.
86. Let  $\mathcal{V}$  be a vector bundle of rank  $n$ . The  $n$ -th exterior power bundle of  $\mathcal{V}$  is called the determinant bundle of  $\mathcal{V}$  and is denoted by  $Det\mathcal{V}$ . Show that  $\mathcal{V}$  is orientable iff  $Det\mathcal{V}$  is trivial.
87. Show that a vector bundle on a path-connected simply connected topological space is always orientable.

88. Let  $B$  be a path-connected, semi-locally simply connected topological space. Suppose that its fundamental group  $\pi_1(B, *)$  does not admit a subgroup of index 2. Then any vector bundle on  $B$  is orientable.
89. Let  $B$  be a differentiable manifold. Suppose that its fundamental group is finite of odd order. Show that  $B$  is orientable.
90. Let  $n$  be odd (resp even). Let  $f(x_0, x_1, \dots, x_n)$  be a homogeneous polynomial of degree  $d$ . Assume that for  $x \neq 0$ ,  $\text{grad } f|_x \neq 0$ . Let  $SV(f)$  resp  $PV(f)$  be the  $(n-1)$ -dimensional spherical (resp) projective varieties defined by  $f$ . Assume that  $S(V(f))$  is non-empty and connected. Show that  $PV(f)$  is orientable iff  $d$  is even (resp odd). In other words,  $PV(f)$  is orientable iff  $n+d \equiv 1 \pmod{2}$ .