Exercises in “Fourier Series and Functional Analysis”

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Sources:

[**B-N-Be**] George Bachman, Lawrence Narici and Edward Beckenstein, *Fourier and Wavelet Analysis*, Springer-Verlag, New York, 2000.

[**L**] Balmohan V. Limaye, *Functional Analysis*, New Age International (P) Ltd., New Delhi, 2004.

[**W-Z**] Richard L. Wheeden and Antoni Zygmund, *Measure and Integral*, Marcel Dekker Inc., New York, 1977.

Notation: $\mathcal{R}$: Real numbers, $\mathcal{N}$: Natural numbers

In Section **A**, the Fourier coefficients of a function are understood to be with reference to *cosines* and *sines*.

A

1. Show that any unconditionally summable series in $\mathcal{R}$ is absolutely summable.
2. Show that a normed linear space X is Banach if and only if every absolutely summable series in X is summable.

3. Let $\{x_n\}$ be a sequence in a Banach space X. Show that $\Sigma_{n\in\mathcal{N}}\, x_n$ is unconditionally summable if and only if for any $\epsilon > 0$ there exists a finite subset J of $\mathcal{N}$ such that, for any finite subset H of $\mathcal{N}$ for which $J \cap H = \emptyset$, $||\,\Sigma_{n\in H}\, x_n|| \leq \epsilon$.

4. Give an example to show that an unconditionally summable series in a Banach space may not be absolutely summable.

5. Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space $\mathcal{H}$.

   (a) If x is in the closure of the linear span of $\{u_\alpha\}$, then prove that

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n \text{ and } ||x||^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2,$$

   where $\{u_1, u_2, ...\} = \{u_\alpha : \langle x, u_\alpha\rangle \neq 0\}$.

   (b) Prove that the linear span of $\{u_\alpha\}$ is dense in $\mathcal{H}$ if and only if every x in $\mathcal{H}$ has a **Fourier Expansion** as in (a) above if and only if for every x, y in $\mathcal{H}$ the **Parseval Identity**

$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, u_n\rangle\langle u_n, y\rangle$$

   holds, where $\{u_1, u_2, ...\} = \{u_\alpha : \langle x, u_\alpha\rangle \neq 0 \text{ and } \langle y, u_\alpha\rangle \neq 0\}$.

6. Show that the **Haar system** is an orthonormal basis for $L^2([0,1])$.

7. Verify that each of the systems
   (a) $\{1, \cos t, \cos 2t, ...\}$
   (b) $\{\sin t, \sin 2t, ...\}$
   is orthogonal and complete in $L^2([0,\pi])$.

8. Verify that the system $\{\sin t, \sin 3t, \sin 5t, ...\}$ is orthogonal and complete in $L^2([0, \frac{\pi}{2}])$.

9. Show that the series
$$\frac{a_0}{2} + \sum_{n\in\mathcal{N}} a_n \cos nt + \sum_{n\in\mathcal{N}} b_n \sin nt$$
   with real coefficients a_n and b_n may be written in the *amplitude-phase* form
$$\frac{a_0}{2} + \sum_{n\in\mathcal{N}} d_n \cos(nt + \phi_n).$$

10. If a real-valued function f has the Fourier series
$$\frac{a_0}{2} + \sum_{n\in\mathcal{N}} d_n \cos(nt + \phi_n)$$
    of Exercise 9, then show that the Fourier series for $f(t+a)$ is given by
$$\frac{a_0}{2} + \sum_{n\in\mathcal{N}} d_n \cos(nt + [\phi_n + na]).$$

11. Find the Fourier series for the sqaure-wave

$$g(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi; \end{cases}$$

and deduce from it the Fourier series for

$$f(t) = \begin{cases} 4, & -\pi < t < 0 \\ 10, & 0 < t < \pi. \end{cases}$$

12. Find the Fourier series for the following functions:

$$(a)\ f(t) = \begin{cases} 0, & -\pi \le t < 0 \\ t, & 0 \le t \le \pi \end{cases}$$

(b) $f(t) = t$ for $-\pi \le t \le \pi$.

13. Let the function f be p-periodic.
(a) Assuming that f is differentiable, must the derivative f' be periodic?
(b) If f is integrable over all closed intervals $[a, b]$, is $F(x) = \int_a^x f(t)dt$ periodic? What if $\int_a^{a+p} f(t)dt = 0$?

14. For any $f \in L^2([0, 1])$, verify that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{|a_0|^2}{2} + \sum_{n \in \mathcal{N}} (|a_n|^2 + |b_n|^2),$$

where a_n and b_n are the Fourier coefficients of f.

15. Any function f can be written as the sum of an even function $f_e(t) = (f(t) + f(-t))/2$ and an odd

function $f_o(t) = (f(t) - f(-t))/2$. For any $f \in L^2([0,1])$, show that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \frac{1}{\pi}\int_{-\pi}^{\pi} |f_e(t)|^2 dt + \frac{1}{\pi}\int_{-\pi}^{\pi} |f_o(t)|^2 dt.$$

What has this got to do with Pythagoras?

16. If there exists a smallest period $p_0 > 0$ for a function f, then p_0 is called the *fundamental period* of f. Prove that if f is a non-constant continuous periodic function on $\mathcal{R}$, then f has a fundamental period.

17. Find the Fourier
(a) cosine series for $f(t) = \sin t$, $0 \le t \le \pi$,
(b) sine series for $f(t) = c$, a constant, for $0 \le t \le p$.

18. Revise the following notions: Lipschitz Condition (L), Uniform Lipschitz Condition (UL), Piecewise Continuous/Smooth Function (PC/PS), Function of Bounded Variation (BV), Absolutely Continuous Function (AC). Interpret and understand the following implications: (i) UL $\Rightarrow$ BV (ii) AC $\Rightarrow$ BV (iii) PS $\Rightarrow$ BV

19. (Variations of the **Riemann-Lebesgue Lemma**): Let f be a real integrable function on $[a, b]$.
(a) Divide $[a, b]$ into n subintervals of length $\frac{b-a}{n}$, and define $g_n = \pm 1$ on alternate subintervals. Show that,

for f continuous, $\lim_{n\to\infty} \int_a^b f(t)g_n(t)dt = 0$.
(b) Show that if $\{g_n\}$ is a unifomly bounded orthonormal basis for $L^2([0,1])$ over $\mathcal{R}$, then $\lim_{n\to\infty} \int_a^b f(t)g_n(t)dt = 0$.

20. (First Mean Value Theorem for integrals): Let f be increasing on $[a,b]$ and let h be real continuous on $[a,b]$. Prove that

$$\int_a^b h(t)df(t) = h(c)(f(b) - f(a)$$

for some $c \in [a,b]$.

21. (Second Mean Value Theorem for integrals, Version I): Let f be increasing on $[a,b]$ and let h be real continuous on $[a,b]$. Prove that that there exists some $c \in [a,b]$ such that

$$\int_a^b f(t)dh(t) = f(a)\int_a^c dh(t) + f(b)\int_c^b dh(t).$$

22. (Second Mean Value Theorem for integrals, Version II, also known as **Bonnet's MVT**): Let f be nonnegative increasing on $[a,b]$ and let g be real continuous on $[a,b]$. Prove that that there exists some $c \in [a,b]$ such that

$$\int_a^b f(t)g(t)dt = f(b)\int_c^b g(t)dt.$$

23. Let f be a real integrable function on $[-\pi, \pi]$ and let f be extended to $\mathcal{R}$ 2π-periodically. If f is a continuous function of bounded variation on $(a, b) \subset [-\pi, \pi]$, then prove that the Fourier series of f coverges uniformly to f on every closed subinterval $[a + r, b - r],\ r > 0$, of $[a, b]$.

In Section **B**, the n'th Fourier coefficient of $\hat{f}(n)$ of $f \in L^1([-\pi, \pi])$ is understood to be $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$.

B

1. Let $X = \{C([-\pi, \pi]) : f(\pi) = f(-\pi)\}$. Consider $f_0(t) = 1,\ f_1(t) = \cos t$, and $f_2(t) = \sin t$ for $t \in [-\pi, \pi]$. Let $P_n : X \to X$ be a positive linear map for $n = 1, 2, ...$ It is a remarkable result of **Korovkin** that if $P_n(f_j) \to f_j$ uniformly on $[-\pi, \pi]$ for $j = 0, 1, 2$, then $P_n(f) \to f$ uniformly on $[-\pi, \pi]$ for every $f \in X$. Use the result of Korovkin to deduce **Fejer's Theorem**: If f is a continuous function on $[-\pi, \pi]$ such that $f(\pi) = f(-\pi)$, then the sequence of arithmetic means of the partial sums of the Fourier series of f converges to f uniformly on $[-\pi, \pi]$.

2. Use Fejer's theorem stated above to deduce the standard version of the Riemann-Lebesgue Lemma: For $f \in L^1([-\pi, \pi])$, $\hat{f}(n)$ coverges to 0 as $n \to \pm\infty$. If $f \in L^2([-\pi, \pi])$ in particular, how could you reach the same conclusion using the Hilbert space theory?

3. For $f \in L^1([-\pi, \pi])$, show that the series $\Sigma_{n=1}^{\infty} \frac{\hat{f}(n) - \hat{f}(-n)}{n}$ converges in $\mathcal{R}$. (Hint: Consider $g(s) = \int_{-\pi}^{\pi} f(t)dt - \hat{f}(0)s$ and note that g is a continuous function of bounded variation so that its Fourier series converges to it at 0.)

4. If, for $f \in L^1([-\pi, \pi])$, $\hat{f}(n) = 0$ for all $n = 0, \pm 1, \pm 2, ...$, then show that $f(t) = 0$ for almost all $t \in [-\pi, \pi]$. (Hint: Use Fejer' theorem). Deduce the following: For $f \in L^1([-\pi, \pi])$, $\Sigma_{-\infty}^{\infty} |\hat{f}(n)| < \infty$ implies that $f(t) = \Sigma_{-\infty}^{\infty} \hat{f}(n)e^{int}$ for almost all $t \in [-\pi, \pi]$.

5. (An application of the **Uniform Boundedness Principle**) Let $X = \{C([-\pi, \pi]) : f(\pi) = f(-\pi)\}$ with the supremum norm. Prove that there is a dense subset Y of X such that, for every $f \in Y$, the Fourier series of f diverges at 0.† (Hint: Consider $s_m(f) = \Sigma_{n=-m}^{m} \hat{f}(n)$ and verify that $\{||s_m|| : m = 0, 1, 2, ...\}$ is unbounded.)

6. (An application of the **Bounded Inverse Theorem**) Show that there are scalars k_n $(n = 0, \pm 1, \pm 2, ...)$ such that $k_n \to 0$ as $n \to \pm\infty$, but there is no $f \in L^1([-\pi, \pi])$ such that $\hat{f}(n) = k_n$ for $n = 0, \pm 1, \pm 2, ...$

7. (An application of a specialized version of the **Banach-Alaoglu Theorem**) Let k_n $(n = 0, \pm 1, \pm 2, ...)$ be a sequence of scalars. Define $s_m(t) = \Sigma_{n=-m}^{m} k_n e^{int}$, $t \in [-\pi, \pi]$, $m = 0, 1, 2, ...$ and $a_m = \frac{s_0 + ... + s_{m-1}}{m}$, $m = 1, 2, ...$ Let $1 < q \leq \infty$ and assume that the sequence $\{a_m\}$ is bounded in $L^q([-\pi, \pi])$. Prove that there is some $f \in L^q([-\pi, \pi])$ such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt = k_n, \quad n = 0, \pm 1, \pm 2, ...,$$

that is, $\Sigma_{n=-\infty}^{\infty} k_n e^{int}$ is the Fourier series of some $f \in L^q([-\pi, \pi])$.
Show that a corresponding assertion fails for the case $q = 1$. What would be an appropriate modification of the preceding result for the case $q = 1$?[††]

[†] For a concrete example of such a function f, refer to Theorem 12.35 of [**W-Z**].
[††] Refer to Exercise 15-22 of [**L**].