

Lecture Notes in Complex Analysis¹

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May 9, 2007

¹ATML in Complex Analysis at BIM pune, 14th May to 26 May 2007

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Chapter 1

Holomorphic Maps

1.1 Complex Differentiability

Recall that for a real valued function f defined in an open interval, and a point x_0 in the interval, we say f is differentiable at x_0 if the limit of the difference quotient

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. Moreover, this limit is then called the derivative of f at x_0 and is denoted by $\frac{df}{dx}$ or by $f'(x_0)$.

In order to talk about differentiability of a function f at a point x of the domain of f , observe that we needed that the map be defined in an interval around x . Similarly, in case of functions defined on a domain D in \mathbb{C} , we shall need that a disc of radius r around the point under consideration is contained in the domain of f . Just to avoid the necessity of mentioning this condition every time, we introduce the concept of an open set here. (Of course, once introduced, this concept starts playing a far more important role by itself than the purpose for which it has been introduced. For more details refer to section 1.4.)

Definition 1.1.1 A subset $U \subset \mathbb{C}$ is called an open set if for each point $z \in U$, we have $r > 0$ such that the open-ball $B_r(z) \subset U$ where

$$B_r(z) = \{w \in \mathbb{C} : |w - z| < r\}.$$

Let us now consider a complex-valued function f defined in an open subset of \mathbb{C} and define the concept of differentiation with respect to the complex variable. With no valid justification or motivation to do otherwise, we opt for a similar definition of differentiability of f in this case also as in the case of real valued functions of a real variable, as a limit of ‘difference quotients’. All that we need is that these ‘difference quotients’ make sense.

Definition 1.1.2 Let $z_0 \in U$, where U is an open subset of \mathbb{C} . Let $f : A \rightarrow \mathbb{C}$ be a map. Then f is said to be *complex differentiable* (written \mathbb{C} -differentiable at z_0) if the limit on the right hand side of the following formula exists, and in that case we call this limit, the derivative of f at z_0 :

$$\boxed{\frac{df}{dz}(z_0) := \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}}. \quad (1.1)$$

We also use the notation $f'(z_0)$ for this limit and call it *Cauchy derivative* of f at z_0 .

If f is \mathbb{C} -differentiable at each $z \in U$ then we say f is *holomorphic*¹ on U . If f is holomorphic on U , the map $z \mapsto f'(z)$ is called the derivative of f on U and is denoted by f' .

Example 1.1.1 Let us work out the derivative of the function $f(z) = z^n$, where n is an integer, directly from the definition. Of course, for $n = 0$,

¹We caution you that the word ‘holomorphic’ has been used by different authors to mean somewhat different things. Luckily, this is not a serious matter, since we shall see that ultimately they all mean the same thing.

the function is a constant and hence, it is easily seen that it is differentiable everywhere and the derivative vanishes identically. Consider the case when n is a positive integer. Then by binomial expansion, we have,

$$f(z+h) - f(z) = h \left(\binom{n}{1} z^{n-1} + \binom{n}{2} h z^{n-2} + \cdots + h^{n-1} \right).$$

Therefore, we have,

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = n z^{n-1}.$$

This is valid for all values of z . Hence f is differentiable in the whole plane and its derivative is given by $f'(z) = n z^{n-1}$. Next, consider the case when n is a negative integer. We see that the function is not defined at the point $z = 0$. Hence we consider only points $z \neq 0$. Writing $n = -m$ and $f(z+h) - f(z) = \frac{z^m - (z+h)^m}{(z+h)^m z^m}$ and applying binomial expansion for the numerator as above, we again see that

$$f'(z) = -\frac{m}{z^{m+1}} = n z^{n-1}, \quad z \neq 0. \quad (1.2)$$

Remark 1.1.1 As in the case of calculus of 1-real variable, the Cauchy derivative has all the standard properties:

(i) The sum $f_1 + f_2$ of two \mathbb{C} -differentiable functions f_1, f_2 is \mathbb{C} -differentiable and

$$(f_1 + f_2)'(z) = f_1'(z) + f_2'(z).$$

(ii) the scalar multiple of a complex differentiable function f is complex differentiable, and

$$(\alpha f)'(z) = \alpha f'(z).$$

(iii) The product of two complex differentiable functions f, g is again complex differentiable and we have the product rule:

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z). \quad (1.3)$$

Further if $g(z) \neq 0$ then we have the quotient rule:

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}. \quad (1.4)$$

All these properties hold point-wise and therefore, we can replace ‘complex differentiable’ by ‘holomorphic on an open set’, in all of them.

The proof of the following theorem is exactly the same as the proof of the corresponding result for real valued function of a real variable.

Theorem 1.1.1 (The Increment Theorem:) *Let $f : A \rightarrow \mathbb{C}$, $z_0 \in A$, $r > 0$ such that $B_r(z_0) \subset A$. Then f is holomorphic at z_0 iff $\exists \alpha \in \mathbb{C}$, and a set theoretic function $\phi : B_s(0) \setminus \{0\} \rightarrow \mathbb{C}$, ($0 < s < r$) such that for all $h \in B_s(0) \setminus \{0\}$*

$$\boxed{(f(z_0+h) - f(z_0) = h\alpha + h\phi(h); \lim_{h \rightarrow 0} \phi(h) = 0.} \quad (1.5)$$

Proof: For the given f , we simply take

$$\psi(h) := \frac{f(z_0+h) - f(z_0)}{h}, \quad h \in B_r(0) \setminus \{0\}.$$

Then f is holomorphic at z_0 iff $\lim_{h \rightarrow 0} \psi(h)$ exists. In that case, we simply put α equal to this limit and take $\phi(h) = \psi(h) - \alpha$ and observe that $\phi(h) \rightarrow 0$ iff $\psi(h) \rightarrow \alpha$. On the other hand, if there is such a function ϕ and a constant α then clearly, $\lim_{h \rightarrow 0} \psi(h) = \alpha$, and so, f is holomorphic at z_0 and $f'(z_0) = \alpha$. ♠

Remark 1.1.2 We may assume that the error function ϕ in (1.5) is defined on the whole of $B_r(0)$, its value at 0 being completely irrelevant for us. The increment theorem enables one to deal with many tricky situations while dealing with differentiability. As an illustration we shall derive the chain rule for differentiation.

Theorem 1.1.2 (Chain Rule :) Let $f : A \rightarrow \mathbb{C}$, $g : B \rightarrow \mathbb{C}$, $f(A) \subset B$ and $z_0 \in A$. Suppose that $f'(z_0)$ and $g'(f(z_0))$ exist. Then $(g \circ f)'(z_0)$ exists and $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

Proof: Let

$$\left. \begin{aligned} f(z_0 + h) - f(z_0) &= hf'(z_0) + h\eta(h); & \eta(h) \rightarrow 0 \text{ as } h \rightarrow 0; \\ \& \quad g(f(z_0) + k) - g(f(z_0)) &= kg'(f(z_0)) + k\zeta(k); & \zeta(k) \rightarrow 0 \text{ as } k \rightarrow 0 \end{aligned} \right\} (1.6)$$

be as in the increment theorem. Let η, ζ be defined over $B_s(0), B_s(0)$ respectively. Since f the continuity of f at z_0 , it follows that if s is chosen sufficiently small then for all $h \in B_s(0)$, we have $f(z_0 + h) - f(z_0) \in B_r(0)$. Hence we can put $k = f(z_0 + h) - f(z_0)$, in (1.6) to obtain,

$$\begin{aligned} g(f(z_0 + h)) - g(f(z_0)) &= [hf'(z_0) + h\eta(h)][g'(f(z_0)) + \zeta(k)] \\ &= hg'(f(z_0))f'(z_0) + h\xi(h), \end{aligned}$$

where, $\xi(h) = \eta(h)g'(f(z_0)) + (\eta(h) + f'(z_0))\zeta(f(z_0 + h) - f(z_0))$. Observe that as $h \rightarrow 0$, we have, $k = f(z_0 + h) - f(z_0) \rightarrow 0$ and $\zeta(k) \rightarrow 0$. Hence $\xi(h) \rightarrow 0$, as $h \rightarrow 0$. Thus by the increment theorem again, $(g \circ f)'(z_0)$ exists and is equal to $g'(f(z_0))f'(z_0)$ as desired.

Remark 1.1.3 So far, we have considered derivatives of functions at a point $z \in A$ only when the function is defined in a nbd of z . We can try to relax this condition as follows: Thus, if $B \subset \mathbb{C}$ and $f : B \rightarrow \mathbb{C}$, we say f is holomorphic on B if f extends to a holomorphic map $\hat{f} : A \rightarrow \mathbb{C}$ where A is an open subset containing B . However, we can no longer attach a unique derivative to f at points of B in general. With some more suitable geometric assumptions on B , this can be made possible. For instance if B is a closed disc or a closed rectangle, and if $f : B \rightarrow \mathbb{C}$ is differentiable on B , then even at the boundary points of B , the derivative of f is unique. However, this is only for curiosity; we shall never have any opportunity to use such finer treatment in this text.

section Polynomials and Rational Functions

In this section, let us study some simple examples holomorphic functions:

(a) The polynomial functions: As such, the easiest functions to deal with are the constant functions. These are the first examples of holomorphic functions. The identity function $z \mapsto z$, merely denoted by z is also holomorphic. Let n be a non negative integer. By a *polynomial function* $p(z)$ of degree n , we mean a function of the form

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n,$$

$a_i \in \mathbb{C}$, $a_n \neq 0$. Since scalar multiples, sums and products of holomorphic functions are holomorphic, it follows that any polynomial function is holomorphic. Moreover, the (Cauchy derivative) complex derivative of p is easily seen to be given by

$$p'(z) = a_1 + 2a_2z + \cdots + na_nz^{n-1}.$$

Observe that all constant functions $a \neq 0$ have degree 0. The ‘zero’ function is customarily assigned degree $-\infty$, but often it can be assigned any particular degree depending upon the context. All degree 1 polynomials are also referred to as *linear polynomials*. The **fundamental theorem of algebra** (FTA) asserts that every *non-constant* polynomial assumes the value zero, i.e., the equation

$$p(z) = 0$$

has a solution. This is the same as saying *every non constant polynomial has a root*. We have already seen a proof of this theorem in chapter 1. Now observe that if z_1 is a solution of $p(z) = 0$, then as in school algebra, we can perform division by $z - z_1$ and the remainder will be zero, i.e.,

$$p(z) = q(z)(z - z_1).$$

It also follows easily that $\deg q(z) = n - 1$. Thus by repeated application of FTA, we can factorize $p(z)$ completely into linear factors and a constant:

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n), \quad a_n \neq 0. \quad (1.7)$$

We conclude that every polynomial of degree n has n roots. Also if $w \neq z_i$, $i = 1, \dots, n$, it is clear from (1.7) that $p(w) \neq 0$. Hence, $p(z)$ has precisely n roots. Observe that two or more of the roots z_j may coincide. If that is the case, we say, that the corresponding root is a *multiple root* with its *order* or *multiplicity* being equal to the number of times $z - z_j$ is repeated in the factorization (1.7). The factorization is unique up to a permutation of the factors.

Next, we observe that if z_1 is a repeated root then $p'(z_1) = 0$. Indeed if the multiplicity of z_1 is m in $p(z)$ then

$$p(z) = (z - z_1)^m q(z), \quad (q(z_1) \neq 0)$$

which implies that,

$$p'(z) = m(z - z_1)^{m-1}q(z) + (z - z_1)^m q'(z) = (z - z_1)^{m-1}r(z),$$

where $r(z) = mq(z) + (z - z_1)q'(z)$. Since $r(z_1) \neq 0$, it follows that the multiplicity of z_1 in $p'(z)$ is $m - 1$.

As an entertaining exercise, let us prove the following theorem due to Gauss which has a lot of geometric content in it. Recall that if S is a subset of \mathbb{C} , then by *convex hull* of S we mean the set of all elements $\sum t_j S_j$ where the sum is finite, $0 \leq t_j \leq 1$ and $\sum t_j = 1$.

Theorem 1.1.3 Gauss²: *Let $p(z)$ be a polynomial with complex coefficients. Then all roots of $p'(z)$ lie in the convex hull spanned by the roots of $p(z)$.*

²An equivalent version of this has been attributed to Lucas by Ahlfors.

Proof: For any complex number z , since $z\bar{z} = |z|^2$, it follows that for any $z \neq 0$, \bar{z}^{-1} has the same argument as z . Let z_1, \dots, z_n be the roots of $p(z)$, so that $p(z) = \prod_{i=1}^n (z - z_i)$. Then we have,

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^n \frac{1}{z - z_j}.$$

Now suppose that w is a root of $p'(z)$. If $w = z_j$ for some j , there is nothing to prove. So, let $w \neq z_j$ for any j . Then it follows that

$$\sum_{j=1}^n \frac{1}{w - z_j} = 0 = \sum_{j=1}^n \frac{1}{\overline{w - z_j}}. \quad (1.8)$$

On the other hand suppose w did not belong to the convex hull of $\{z_1, z_2, \dots, z_n\}$, then it is easily seen that there is a straight line L passing through w such that all the points z_j lie strictly to one side of L . (Write full details as an exercise.) If L_1 is the line through the origin parallel to L , then it will mean that all the numbers $w - z_j$ lie on one side of L_1 . Since $(\overline{w - z_j})^{-1}$ have the same argument as $w - z_j$, it follows that $(\overline{w - z_j})^{-1}$ also lie on the same side of L_1 . But then, their sum cannot be zero! This contradiction to (1.8) proves the result. ♠

Remark 1.1.4 One can think of n forces of magnitude $|w - z_j|^{-1}$ acting on the point w and directed towards the point z_j . Then (1.8) can be interpreted as saying that the point w is at equilibrium under these forces. From this interpretation, the conclusion of the theorem is immediate for a physicist. That is how Gauss may have discovered this result. We have deliberately left out a few details in the above proof. These details should be supplied by the reader.

(b) The rational functions: In (a), the domain of definition of our functions were not mentioned. However, note that all those functions were defined throughout \mathbb{C} .

We shall now define some holomorphic functions with their domains of definition not necessarily being the entire plane. These are functions of the form

$$\phi(z) = \frac{p(z)}{q(z)} \quad (1.9)$$

where p and q are polynomials, called *rational functions*. By canceling out common factors from both p and q , we can assume that p and q do not have any common factors. Then, obviously, $\phi(z)$ makes sense only when $q(z) \neq 0$ and so the domain of definition of $\phi(z)$ is $\mathbb{C} \setminus \{z : q(z) = 0\}$. We have by 1.4,

$$\phi'(z) = \frac{p'(z)q(z) - q'(z)p(z)}{(q(z))^2} \quad (1.10)$$

and so ϕ is holomorphic in its domain. Its derivative is another rational function having the same domain of definition. *Prove this statement. Caution: (1.10) may not be in the reduced form even though (1.9) is.* Later on, we shall have more opportunities to study these functions, particularly, the so called *fractional linear transformations*, which are of the form $\frac{az + b}{cz + d}$. At this stage, it may be worthwhile to note that the set of all polynomials in one variable with complex coefficients forms a commutative ring which we denote by $\mathbb{C}[z]$. One of the important property of this ring is that it is an *integral domain*], (i.e., a commutative ring in which product of two non zero elements is never zero. This follows easily from the ‘unique factorization’ property (1.7) that we have seen. The set of all rational functions forms a field, $\mathbb{C}(z)$, called the *field of fractions* of the integral domain $\mathbb{C}[z]$.

In order to get any other class of examples of complex differentiable functions, we have to use the power series. This topic will be taken up soon by

Prof. K. D. Joshi.

Let us now try to understand the complex differentiability a little more closely.

1.2 Cauchy–Riemann Equations

Definition 1.2.1 Let U be an open subset of \mathbb{C} , $z_0 = (x_0, y_0) \in U$ and $f : U \rightarrow \mathbb{C}$ be a given function. By keeping the variable y constant at $y = y_0$ and varying only x , we obtain a function of one variable out of f . More precisely, choose $\epsilon > 0$, so that for $|t| < \epsilon$, the points $(x_0 + t, y_0)$ are inside U . Let

$$F(t) = f(x_0 + t, y_0). \quad (1.11)$$

Then F is a function of a real variable t . Of course, it may be a complex valued function though. Nevertheless, we can talk about differentiability of this function at the point 0. We say the *partial derivative of f with respect to the variable x* exists at z_0 iff F is differentiable at 0 and in this case we set $F'(0)$ to be the partial derivative of f with respect to x . This is denoted by $f_x(z_0)$ or $\frac{\partial f}{\partial x}(z_0)$.

Observe that F is nothing but the restriction of f to the line through z_0 parallel to the x -axis. Similarly, by taking the function f restricted to line through z_0 , parallel to the y -axis, the partial derivative with respect to y is also defined and denoted by f_y or $\frac{\partial f}{\partial y}(z_0)$.

Write $z = x + iy$, $z_0 = x_0 + iy_0$, $h = h_1 + ih_2$, $f(x + iy) = u(x, y) + w(x, y)$. Suppose $f'(z_0)$ exists and let $f'(z_0) = \alpha + i\beta$. By the increment theorem, we have

$$f(z_0 + h) - f(z_0) = hf'(z_0) + h\eta(h); \quad \eta(h) \rightarrow 0 \text{ as } h \rightarrow 0. \quad (1.12)$$

Put $\eta(h) = \eta_1(h) + i\eta_2(h)$, where η_1, η_2 are real and imaginary parts of η . Then clearly,

$$\eta_j(h) \longrightarrow 0 \text{ as } h \longrightarrow 0. \quad (1.13)$$

Put $h = h_1$ in (1.12), i.e., take $h_2 = 0$, to obtain

$$f(x_0 + h_1, y_0) - f(x_0, y_0) = h_1(\alpha + i\beta) + h_1\eta(h_1). \quad (1.14)$$

Comparing the real and the imaginary parts on both the sides we get

$$\left. \begin{aligned} u(x_0 + h_1, y_0) - u(x_0, y_0) &= h_1\alpha + h_1\eta_1(h_1); \\ \text{and } v(x_0 + h_1, y_0) - v(x_0, y_0) &= h_1\beta + h_1\eta_2(h_1). \end{aligned} \right\} \quad (1.15)$$

Therefore, by the increment theorem for 1-variable functions, it follows that u_x and v_x exist at (x_0, y_0) and we have,

$$u_x(x_0, y_0) = \alpha, \quad v_x(x_0, y_0) = \beta. \quad (1.16)$$

Now put $h = ih_2$, i.e., take $h_1 = 0$ in (1.12), to obtain

$$f(x_0, y_0 + h_2) - f(x_0, y_0) = ih_2(\alpha + i\beta) + ih_2\eta(ih_2)$$

Again, comparing the real and imaginary parts and using increment theorem, we obtain

$$u_y(x_0, y_0) = -\beta; \quad v_y(x_0, y_0) = \alpha. \quad (1.17)$$

Thus (1.16) and (1.17) together give

$$u_x(x_0, y_0) = v_y(x_0, y_0); \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \quad (1.18)$$

These are called Cauchy–Riemann(CR)-equations.³ Observe that we also have.

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \quad (1.19)$$

$$= v_y(x_0, y_0) - iv_u(x_0, y_0) \quad (1.20)$$

³Bernhard Riemann(1826-1866) a German mathematician.

and

$$\boxed{|f'(z_0)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2 = u_x v_y - u_y v_x.} \quad (1.21)$$

The last expression above, which is the determinant of the matrix

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \quad (1.22)$$

is called the *Jacobian* of the mapping $f = (u, v)$, with respect to the variables (x, y) and is denoted by

$$\boxed{J_{(x,y)}(u, v) := u_x v_y - u_y v_x.} \quad (1.23)$$

Remark 1.2.1 A simple minded application of CR-equations is that it helps us to detect easily when a function is **not** \mathbb{C} -differentiable. For example, $\Re(z)$, $\Im(z)$ etc are not complex differentiable anywhere. The function $z \mapsto |z|^2$ is not complex differentiable for any point $z \neq 0$. However, it satisfies the CR-equations at 0. That of course does not mean that it is \mathbb{C} -differentiable at 0. (See the exercise below.)

In order to understand the full significance of CR-equations, we must know a little more about calculus of two real variables. In the next section, we recall some basic results in real multi-variable calculus. and then study the close relationship between complex differentiation and the real total differentiation. You may choose to skip this section and come back to it if necessary while reading the section after that. However, try all the following exercises before going further. If you have difficulty in solving any of them, then perhaps you must read the next section thoroughly.

Exercise 1.2.1

1. Use CR equations to show that $z \longrightarrow \Re(z)$, $z \longrightarrow \Im(z)$ are not complex differentiable anywhere.

2. Use polar coordinates to show that $z \mapsto |z|^2$ is complex-differentiable at 0. What about the function $z \mapsto |z|$?
3. Suppose f is a complex differentiable everywhere on an open disc D and takes real values only. Then show that f is a constant. [Hint Use CR equations.] Let now $f : D \rightarrow \mathbb{C}$ be a complex differentiable function. Suppose its image is contained in a line or a circle or a parabola. Then prove that f is a constant.

1.3 Review of Calculus of Two Real Variables

We will need some basic notions of the calculus of two real variables. In this section, we recall these concepts to the extent required to understand the later material that we are going to learn. Indeed, we presume that you are already reasonably familiar with the material of this section.

As a warm-up, we illustrate the kind of danger that we may be in while we are trying to relate the calculus of several variables to that of 1-variable, with an example.

Example 1.3.1 Define a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases} \quad (1.24)$$

This function has questionable behavior only at $(0, 0)$. It has the property that for each fixed y , it is continuous for all x and for each fixed x it is continuous for all y . Moreover, if you restrict the function to any line, it is continuous. However, it is not continuous at $(0, 0)$ even if we are ready to redefine its value at $(0, 0)$. This is checked by taking limits along the line $y = mx$. For different values of m , we get different limits at $(0, 0)$. So, there is no way we can make it continuous at $(0, 0)$.

We begin by recalling the increment theorem of 1-variable calculus.

Theorem 1.3.1 *Let $f : (a, b) \rightarrow \mathbb{R}$ be a function $x_0 \in (a, b)$. Then f is differentiable at x_0 iff there exists an error function η defined in some neighborhood $|x - x_0| < \epsilon$ of x_0 and a real number α such that*

$$\boxed{f(x_0 + h) - f(x_0) = \alpha h + \eta(h)h, \text{ and } \eta(h) \rightarrow 0, \text{ as } h \rightarrow 0.} \quad (1.25)$$

Remark 1.3.1 The proof of this is exactly same as that of theorem 1.1.1. Roughly speaking, the condition in the increment theorem tells us that the difference (increment) in the functional value of f at x_0 is $f'(x_0)$ times the difference (increment) h , in the variable x , up to a second order term viz., $h\eta(h)$. That explains why this result is called the increment theorem. It is also referred to as *linear approximation* to f and is written in the form

$$f(x_0 + h) \approx f(x_0) + hf'(x_0)$$

We know that $f'(x_0)$ is the slope of the tangent to the graph of the function $y = f(x)$. We also know that a line in \mathbb{R}^2 is the graph of a linear map. Since the tangent line represents an approximation of the graph of the function f , we may say that the linear map corresponding to the tangent line represents an approximation to the function f at x_0 . Thus we see that the derivative should be thought of as a *linear map* approximating the given function at the given point. This aspect of the differentiability of a 1-variable function is obscured by the over simplification that occurs naturally in 1-variable linear algebra viz., ‘a linear map $\mathbb{R} \rightarrow \mathbb{R}$ is nothing but the multiplication by a real number and thus can be identified with that real number’. When we pass to two or more variables, this simplification disappears and thus the true nature of the derivative comes out, as in the following definition. In what follows, we restrict our attention to two variables, though there logical gain in it. All the concepts and results that we are going to introduce for two variables hold good for more number of variables also.

Definition 1.3.1 Let U be an open subset of \mathbb{R}^2 and let $f : U \rightarrow \mathbb{R}$ be any function. Let z_0 be any point in U . We say f is (*Frechet*⁴) *differentiable* at z_0 iff there exists a linear map $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ and a scalar valued error function η defined in a neighborhood of z_0 in U such that

$$\boxed{f(z_0+h) - f(z_0) = L(h) + \|h\|\eta(h); \eta(h) \rightarrow 0 \text{ as } h \rightarrow 0.} \quad (1.26)$$

Further L is called the *Frechet derivative* or the *total derivative* of f at z_0 and is denoted by $(Df)_{z_0}$. If f is differentiable at each point of U then it is called a (*Frechet*) *differentiable function*.

As an easy exercise prove the following theorem:

Theorem 1.3.2 *If f is differentiable at a point then it is continuous at that point.*

Theorem 1.3.3 *Let U , f , $z_0 = (x_0, y_0)$ etc. be as above. Let f be Frechet differentiable at z_0 . The f has its partial derivatives at z_0 and moreover we have*

$$f_x(z_0) = (Df)_{z_0}(1, 0); \quad f_y(z_0) = (Df)_{z_0}(0, 1).$$

Proof: Let $L = (Df)_{z_0}$. Putting $h = (t, 0)$ in (1.26), we obtain,

$$F(t) - F(0) = L(t, 0) + \eta(t, 0)|t| = tL(1, 0) + \eta(t, 0)|t|. \quad (1.27)$$

Dividing out by t and taking limit as $t \rightarrow 0$, it follows that $F'(0)$ exists i.e., $f_x(z_0)$ exists and is equal to $L(1, 0)$. Similarly, we can show that f_y exists and $f_y(z_0) = L(0, 1)$. ♠

Remark 1.3.2

(i) In (1.27), can you rewrite the rhs in form of (1.5)? If so, you don't have

⁴René Maurice Frechet (1878-1973).

to divide by t and take the limit etc. as in we did in the proof above.

(ii) The concept of partial derivative is a special case of a more general concept. Given any unit vector \mathbf{u} , we define the *directional derivative* of f in the direction of \mathbf{u} denoted by $D_{\mathbf{u}}f(z_0)$, to be the limit of

$$\lim_{t \rightarrow 0} \frac{f(z_0 + t\mathbf{u}) - f(z_0)}{t}$$

provided it exists. As above, it can be seen that all the directional derivatives exist if $(Df)_{z_0}$ exists. Moreover, by putting $h = t\mathbf{u}$, in (1.26), dividing out by t and then taking the limit as $t \rightarrow 0$, it is verified that $D_{\mathbf{u}}f(z_0) = L(\mathbf{u})(Df)_{z_0} \cdot \mathbf{u}$, where \cdot denotes the dot product.

(iii) It may happen that all the directional derivatives exist and yet the total derivative $(Df)_{z_0}$ may not exist. This can happen even if all this directional derivatives vanish, as seen in the following two examples.

Example 1.3.2 Consider the function

$$f(x, y) = \begin{cases} \frac{x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Clearly f is differentiable at all points except perhaps at $(0, 0)$. We shall show that f is not even continuous at $(0, 0)$ and hence cannot be differentiable at $(0, 0)$. However, observe that if you restrict the function to any of the lines through the origin, then it is continuous. This will tell you that if we approach the origin along any of these lines then the limit of the function coincides with the value of the function. In contrast, in the case of 1-variable function, if the left-hand and right-hand limits existed and agreed with the functional value then the function was continuous at that point. Thus, the geometry of the plane is not merely the geometry of all the lines in it.

In order to see that the function is not continuous at the origin, we shall produce various sequences $\{u_n\}$ such that $\lim_{n \rightarrow \infty} u_n = (0, 0)$ and

$\lim\{f(u_n)\}$ takes different values. It then follows that, at $(0,0)$, even a redefinition of f will not make it continuous. So take any real sequence $x_n \rightarrow 0$, $x_n \neq 0$ and put $y_n = kx_n^2$ for some real k . Put $u_n = (x_n, y_n)$. Then $u_n \rightarrow (0,0)$ and $f(u_n) = k/(1+k^2)$. Therefore, $\lim_{n \rightarrow \infty} f(u_n) = k/(1+k^2)$. Thus for different values of k we get different values of this limit as required.

On the other hand, let $\mathbf{u} = (a, b)$ be a unit vector. If b is zero then clearly the partial derivative of f in the direction of \mathbf{u} (it is f_x) is zero since the function is identically zero on the x axis. For $b \neq 0$, we have $F_{\mathbf{u}}(t) = a^2bt/(t^2a^4 + b^2)$ for all t . It follows that $D_{\mathbf{u}}f(0,0) = F'_{\mathbf{u}}(0) = a^2b/b^2 = a^2/b$. Thus all the directional derivatives exist. Also, for your own satisfaction check that the partial derivatives are not continuous at $(0,0)$.

Example 1.3.3 We can improve upon the above example 1.3.2, as follows. Take $g(x, y) = \sqrt{x^2 + y^2}f(x, y)$, where f is given as in example 1.3.2. Then the function g is continuous also at $(0,0)$ and has all the directional derivatives vanish at $(0,0)$. That means that the graph of this function has the xy -plane as a plane of tangent lines at the point $(0,0,0)$. We are tempted to award such ‘nice’ geometric behavior of the function and admit it to be ‘differentiable’ at $(0,0)$. Alas! Even then it is not differentiable at $(0,0)$, in the definition that we have adopted. For

$$\frac{g(x, y) - g(0, 0)}{\|(x, y)\|} = f(x, y)$$

has no limit at $(0,0)$.

We hope that the above two examples illustrate the subtlety of the situation in the following theorem, which is a result in the positive direction.

Theorem 1.3.4 *Let U be an open set in \mathbb{C} , and $f : U \rightarrow \mathbb{R}$ be a function having partial derivatives which are continuous at (x_0, y_0) . Then f is Frechet differentiable at (x_0, y_0) .*

Proof: By Mean Value theorem of 1-variable calculus, there exist $0 \leq t, s \leq 1$ such that $f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) = kf_y(x_0 + h, y_0 + sk)$; $f(x_0 + h, y_0) - f(x_0, y_0) = hf_x(x_0 + th, y_0)$. (Of course t, s depend on h, k .) Therefore,

$$\begin{aligned} & |f(x_0 + h, y_0 + k) - f(x_0, y_0) - hf_x(x_0, y_0) - kf_y(x_0, y_0)| \\ & \leq |h||f_x(x_0 + th, y_0) - f_x(x_0, y_0)| + |k||f_y(x_0 + h, y_0 + sk) - f_y(x_0, y_0)|. \end{aligned}$$

By continuity of f_x, f_y at (x_0, y_0) given $\epsilon > 0$, we can choose h, k sufficiently small so that $|f_x(x_0 + th, y_0) - f_x(x_0, y_0)| < \epsilon$; $|f_y(x_0 + h, y_0 + sk) - f_y(x_0, y_0)| < \epsilon$. The conclusion follows. ♠

A slight variation of the above result is given below. The proof is left to you as a simple exercise.

Theorem 1.3.5 *Let U be an open subset of \mathbb{C} and $f : U \rightarrow \mathbb{R}$ be a function. Then f is Frechet differentiable at U and the function $f' : U \rightarrow \mathbb{C}$ is continuous iff both the partial derivatives of f exist on U and are continuous on U .*

Observe that the assignment $\mathbf{x} \mapsto (Df)_{\mathbf{x}}$ defines a map Df of U into the space of all linear maps \mathbb{R}^2 into \mathbb{R} viz., the dual vector space \mathbb{R}^{2*} which is isomorphic to \mathbb{R}^2 . So, one can define f to be *continuously differentiable* if Df is defined and continuous. Since the two coordinate functions of Df are nothing but the two partial derivatives, the continuity of Df is equivalent to that of the continuity of the two partial derivatives of f .

Of course, if this is true for all points $\mathbf{x} \in U$ then we say f is *continuously differentiable* in U or f is of *class \mathcal{C}^1* in U . Inductively, a function f on U is said to be of *class \mathcal{C}^r* in U if all the partial derivatives of f of order r exist and are continuous in U . Finally, a function which is of class \mathcal{C}^r for all $r > 0$ is said to be of class \mathcal{C}^∞ .

Also observe that in the case of one variable, we get back our old definition provided we identify $(Df)_{z_0}$ with $L(1)$.

Remark 1.3.3 All the standard properties of the derivatives of a function of one variable such as for sums and scalar multiples etc. hold here also with obvious modifications wherever necessary. For instance, if f and g are real valued differentiable functions then their product is differentiable and we have

$$D(fg)(z_0) = f(z_0)D(g)(z_0) + g(z_0)D(f)(z_0).$$

It may be worth recalling that Mean Value Theorem is one result which really needs modification.

Definition 1.3.2 Let $f : U \rightarrow \mathbb{R}^2$ be a function and z_0 be a point of the open set $U \subset \mathbb{C}$. We say f is differentiable at z_0 there exists a linear map $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and an error function $\eta : B_r(0) \rightarrow \mathbb{R}^2$ such that for $h \in B_r(0)$, we have,

$$\boxed{(f(z_0 + h) - f(z_0) = L(h) + |h|\eta(h); \lim_{h \rightarrow 0} \eta(h) = 0.} \quad (1.28)$$

In this case, We write $D(f)_{z_0} = L$.

Theorem 1.3.6 Let $f : U \rightarrow V$ and $g : V \rightarrow \mathbb{R}$ be such that f is differentiable at $z_0 \in U$ and g is differentiable at $w_0 = f(z_0) \in V$. Then $g \circ f$ is differentiable at z_0 and we have,

$$D(g \circ f)_{z_0} = D(g)_{w_0} \circ D(f)_{z_0}.$$

Proof: We have

$$f(z_0 + h) - f(z_0) = L_1(h) + |h|\eta_1(h); \quad g(w_0 + k) - g(w_0) = L_2(k) + |k|\eta_2(k),$$

where $\eta_1(h) \rightarrow 0$ as $h \rightarrow 0$ and $\eta_2(k) \rightarrow 0$ as $k \rightarrow 0$. Note that $f(z_0 + h) - f(z_0) \rightarrow 0$ as $h \rightarrow 0$. Therefore, we can substitute $k = f(z_0 + h) - f(z_0)$ in the second equation. This gives,

$$\begin{aligned}
g \circ f(z_0 + h) - g \circ f(z_0) &= L_2(L_1(h) + |h|\eta_1(h)) + |h| \left(\frac{\|k\|}{|h|} \eta_2(k) \right) \\
&= L_2 \circ L_1(h) + |h| \left(L_2(\eta_1(h)) + \frac{\|k\|}{|h|} \eta_2(k) \right)
\end{aligned}$$

Observe that

$$\frac{\|k\|}{|h|} \leq \frac{\|L_1(h)\|}{|h|} + \|\eta_1(h)\| \longrightarrow \|L_1\|.$$

Therefore, if we take $\eta(h) = L_2(\eta_1(h)) + \frac{\|k\|}{|h|} \eta_2(k)$, it follows that $\eta(h) \longrightarrow 0$ as $h \longrightarrow 0$. The result follows. ♠

As an easy consequence, we can now derive:

Theorem 1.3.7 *Let U be a convex open subset of \mathbb{R}^2 and $f : U \longrightarrow \mathbb{R}^2$ be a differentiable function such that $D(f)_z = 0$ for all $z \in U$. Then $f(z) = c$, a constant, on U .*

Proof: Fix a point $z_0 \in U$. Now for any point $z \in U$ consider the map $g : [0, 1] \longrightarrow U$ given by $g(t) = (1-t)z_0 + tz$. By chain rule the composite map $h := f \circ g : [0, 1] \longrightarrow \mathbb{R}^2$ is differentiable and its derivative vanishes everywhere. By 1-variable calculus, (Lagrange’s Mean Value theorem), applied to each component of $h = (h_1, h_2)$ it follows that h is a constant function on $[0, 1]$. In particular, $h(1) = h(0)$. But $f(z) = h(1) = h(0) = f(z_0)$. ♠

Remark 1.3.4 Observe that the projection maps are differentiable. Therefore, it follows that if $f = (f_1, f_2)$ is differentiable then each co-ordinate function f_j is also so. It is not difficult to see that the converse is also true. The derivatives $D(f_1)$ and $D(f_2)$ can be treated as row vectors and by writing them one below the other, we get a 2×2 matrix $D(f)$. With this notation, the chain rule can be stated in terms of matrix multiplication.

Having identified a linear map $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with a 2×2 real matrix, we see that $D(f)$ is a function from U to $M(2; \mathbb{R})$. The latter space can

be identified with the Euclidean space \mathbb{R}^4 . We can then see that $D(f) : U \rightarrow \mathbb{R}^4$ is continuous iff the partial derivatives of f_1, f_2 are continuous. The map $f : U \rightarrow \mathbb{R}^2$ is called a map of class \mathcal{C}^1 on U if it is differentiable and the derivative $D(f)$ is continuous. (This also goes under the somewhat loose terminology ‘continuously differentiable’.) What is then the meaning of $D(f) : U \rightarrow \mathbb{R}^4$ is differentiable? Going by the above principle, we see that this is the same as saying that all the four component functions of $D(f)$ should be differentiable. The derivative of $D(f)$ is actually a function $D^2(f) : U \rightarrow \mathbb{R}^8$. Components of this are nothing but the second order partial derivatives of the components of f . Thus for any positive integer k , we define f to be of class \mathcal{C}^k on U if all its k -th order partial derivatives exist and are continuous on U . If f is of class \mathcal{C}^k for all $k \geq 1$ then it is said to belong to the class \mathcal{C}^∞ . Such maps are also called smooth maps.

All this can be easily generalized to functions from subsets of \mathbb{R}^n to \mathbb{R}^m for any positive integers m, n .

1.4 Cauchy Derivative (Vs) Frechet Derivative

Partial derivatives play a key role in the comparison study of Cauchy derivative and Frechet derivative. We have seen that existence of either of them implies the existence of partial derivatives. Moreover, in the former case, the partial derivatives satisfy the CR-equations. Thus, even if Df exists, if CR equations are not satisfied then f' does not exist. Using this we can give plenty of examples of non-holomorphic functions which are Frechet differentiable. As we have already seen, the geometry of the plane is responsible for making the total derivative somewhat subtler in comparison with the derivative in the case of one 1-variable function. What additional basic structure

is then responsible for the difference in Cauchy differentiation and Frechet differentiation? An answer to this question is in the following theorem:

Theorem 1.4.1 *Let $f : U \rightarrow \mathbb{C}$ be a continuous function, $f = u + v$, $z_0 = x_0 + iy_0$ be a point of U . Then f is \mathbb{C} -differentiable at z_0 iff considered as a vector valued function of two real variables, f is (Frechet) differentiable at z_0 and its derivative $(Df)_{z_0} : \mathbb{C} \rightarrow \mathbb{C}$ is a complex linear map. In that case, we also have $f'(z_0) = (Df)_{z_0}$.*

Proof: Recall that a map $\phi : V \rightarrow W$ of complex vector spaces is complex linear iff $\phi(\alpha v + \beta w) = \alpha\phi(v) + \beta\phi(w)$ for any $\alpha \in \mathbb{C}$ and $v, w \in V$. Let us first consider a purely algebraic problem: Treating \mathbb{C} as a 2-dimensional real vector space, consider a real linear map $T : \mathbb{C} \rightarrow \mathbb{C}$ given by the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

When is it a complex linear map? We see that, if T is complex linear, then $T(i) = iT(1)$ and hence, $b + id = T(i) = iT(1) = i(a + ic)$. Therefore, $b = -c$ and $a = d$. Conversely, it is easily seen that this condition is enough to ensure the complex linearity of T .

Coming to the proof of the theorem, suppose that f is \mathbb{C} -differentiable at z_0 . Then as already seen the partial derivatives exist and satisfy the Cauchy–Riemann equations. So, the 2×2 matrix (1.22) defines a complex linear map from \mathbb{C} to \mathbb{C} . It remains to see that f is real differentiable at z_0 , for then, automatically the derivative will be equal to the matrix (1.22) above. For this, we directly appeal to the increment theorem: We have,

$$f(z_0 + h) - f(z_0) = (\alpha + i\beta)h + h\eta(h),$$

where, $\alpha = u_x = v_y$, $\beta = v_x = -u_y$. Put $\phi(h) = \frac{h}{\|h\|}\eta(h)$. Then,

$$\lim_{h \rightarrow 0} \|\phi(h)\| = \lim_{h \rightarrow 0} \|\eta(h)\| = 0.$$

Also, the multiplication map $h \mapsto (\alpha + \beta i)h$ can be viewed as a real linear map acting on the 2-vector h , it follows that f is Frechet differentiable, with the derivative Df given by (1.22).

Conversely, assume that Df exists and is complex linear. The existence of Df means that there exist error functions ζ and γ say, such that

$$\left. \begin{aligned} u(z_0 + h) - u(z_0) &= (h_1 u_x + h_2 u_y) + |h|\zeta(h) \\ v(z_0 + h) - v(z_0) &= (h_1 v_x + h_2 v_y) + |h|\gamma(h) \end{aligned} \right\} \quad (1.29)$$

with $\zeta(h) \rightarrow 0$ and $\gamma(h) \rightarrow 0$ as $h \rightarrow 0$. Complex linearity of Df means that $u_x = v_y := \alpha$ say, $u_y = -v_x := \beta$ say. Then, multiply the second equation by i and add it to the first equation above to obtain

$$f(z_0 + h) - f(z_0) = h(\alpha + i\beta) + |h|(\zeta(h) + i\gamma(h)),$$

where,

$$\lim_{h \rightarrow 0} \frac{|h|(\zeta(h) + i\gamma(h))}{h} = 0.$$

Finally, in this case, the complex linear map $D(f)_{z_0}$ given by the matrix (1.22) is nothing but multiplication by the complex number $u_x + iu_y = f'(z_0)$. Hence, the proof of the theorem is complete. \spadesuit

As an immediate corollary, we have,

Theorem 1.4.2 *Let $f : U \rightarrow \mathbb{C}$ be a complex differentiable function in a convex domain U such that $f'(z) \equiv 0$. Then f is a constant on U .*

Proof: By the previous theorem the Frechet derivative of f vanishes in U and so we can apply theorem 1.3.7. \spadesuit

There are certain useful partial results that relate the two notions of differentiability. We shall mention some of them here without proof. The most popular one is:

Theorem 1.4.3 *Let f be a continuous complex valued function of a complex variable defined on an open subset U , possessing continuous partial derivatives. Then f is complex differentiable in U iff it satisfies the Cauchy–Riemann equations in U .*

Proof: The only if part has been proved already. The if part is the consequence of theorems 1.3.5 and 1.4.1. along with the observation that CR-equations are equivalent to say that the Frechet derivative is complex linear.

We can improve upon this by:

Theorem 1.4.4 *Let f be a continuous complex valued function of a complex variable defined on an open subset U . Then f is complex differentiable in U iff it has continuous partial derivatives in U which satisfy CR equations.*

Remark 1.4.1 In the ‘only if’ part, the only thing that is not proved already is the continuity of the partial derivatives. We shall not prove it here. It will follow once we show that any Cauchy differentiable function has a continuous derivative. Indeed, we shall see later on that complex differentiability ensures that the function has continuous derivatives of all orders!

The next step is to remove even the continuity hypothesis on the partial derivatives. This has to be done carefully as in the following result known as **Looman-Menchoff Theorem** It is the most general result known in this direction in which the continuity hypothesis on the partial derivatives is removed. However, observe that this is not a ‘pointwise statement’. The proof involves ideas that are beyond the theme of this course. Interested reader can look in [N].

Theorem 1.4.5 *Let U be an open subset of \mathbb{C} and $f : U \rightarrow \mathbb{C}$ be a continuous function, $f = u + iv$. Suppose the partial derivatives of u, v exist and satisfy Cauchy-Riemann equations throughout U . Then f is holomorphic in U .*

Remark 1.4.2 There are many functions that are complex differentiable at a point but not so at any other points in a neighborhood (see the exercises below). As far as the differentiation theory is concerned such functions are not of much use to us. We would like to concentrate on those functions which are differentiable in some non empty open subset of \mathbb{C} . Such functions will be called '*holomorphic*.' Once again, we emphasize the fact functions which have first order partial derivatives satisfying C-R equations in a non empty open subset of \mathbb{C} are holomorphic. However, in practice, using C-R equations to see whether a function is holomorphic or not, would be the last thing that we would like to do. We should have a large class of holomorphic functions readily known to us and then often a new one could be expressed in some nice way in terms of these known ones.

Chapter 2

General Form of Cauchy's Theorem

2.1 Homotopy: Simple Connectivity

On a simple open arc, there is ‘essentially’ only one way to go from one point to another. In contrast, on a circle, there are at least two different ways to do this. As we have already seen, one can interpret the word ‘to go’ here to mean ‘to communicate’ or ‘to connect by a path’. Thus the first case could be referred to as ‘simple connectivity’ and the later as ‘multi-connectivity’. This is how the originators of this notion must have thought as the words used by them indicate. In modern times, these notions are made to work in a larger context and hence a certain abstract, more rigorous and (hence) dry definitions have been adopted in the study of Algebraic Topology using the machinery of the fundamental group. We shall not take full recourse to that here, whereas we shall introduce the concept of homotopy and ‘correct’ modern definition of simply connectedness. Classically the approach for simple connectivity came through the properties of integrals on them, and we shall refer to this by homological simple connectivity.

Definition 2.1.1 Let $\omega_j : [0, 1] \rightarrow \Omega$, $j = 0, 1$, be any two paths with the same initial and terminal points:

$$\omega_0(0) = \omega_1(0) = a; \omega_0(1) = \omega_1(1) = b.$$

We say ω_0, ω_1 are path homotopic to each other in Ω and express this by writing $\omega_0 \sim \omega_1$ if there exists a continuous map $H : I \times I \rightarrow \Omega$ such that

$$H(t, j) = \omega_j(t); H(j, s) = \omega_j(0), \quad j = 0, 1, \quad 0 \leq t \leq 1, \quad 0 \leq s \leq 1.$$

H is called a path-homotopy from ω_0 to ω_1 .

If $a = b$, that is when both the paths are loops passing through a , the above path-homotopy gives a 'loop homotopy' of loops based at a . If ω_0 happens to be the constant loop, we say the loop ω_1 is 'null-homotopic.'

The importance of this notion lies in the following theorem:

Theorem 2.1.1 Homotopy Invariance of Integrals *Let f be a holomorphic mapping on a domain Ω , and ω_j be any two (continuous) contours in Ω which are path homotopic in Ω . Then*

$$\int_{\omega_0} f dz = \int_{\omega_1} f dz.$$

Proof: The idea is that the homotopy H defines a 'continuous family' $\{\omega_s : 0 \leq s \leq 1\}$ of paths beginning with ω_0 and ending with ω_1 and having the same end points. The claim is that for all these paths the integral $\int_{\omega_t} f dz$ takes the same value. Unfortunately, even to make sense out of this claim there is a technical snag: the intermediary paths $\omega_s, 0 < s < 1$ may not be piecewise smooth. Let us agree for a moment that we can handle this, not so serious a snag.

By compactness, by choosing t, s very close, we can make the entire paths ω_t, ω_s to be very close to each other. Fix two such t and s and denote by

$\mu = \omega_t, \nu = \omega_s$. We can then cover both of them by a finite number of discs D_1, D_2, \dots, D_n such that these discs form a chain viz., $D_i \cap D_{i+1} \neq \emptyset$. We can then choose a sequence of points $a = a_0, \dots, a_n = b$; on μ and $a = b_0, \dots, b_n = b$ on ν so that $a_i, b_i \in D_i \cap D_{i+1}$. Let μ_i denote the portion of the contour μ from a_i to a_{i+1} . Similarly define ν_i also. For any two points $z, w \in \mathbb{C}$, let $[z, w]$ denote the line segment traced from z to w . Let P_i denote the closed contour $[a_i, b_i] \star \nu_{ij} \star [b_{i+1}, a_{i+1}]$. Then for each i the entire closed contour $P_i \star \mu_i^{-1}$ lies inside the disc D_i and hence by Cauchy's theorem, the integral of f along this contour vanishes. Therefore

$$\int_{\mu_i} f dz = \int_{P_i} f dz, i = 0, 1, \dots, n-1. \quad (2.1)$$

Therefore

$$\int_{\mu} f dz = \sum_{i=0}^{n-1} \int_{\mu_i} f dz = \sum_{i=0}^{n-1} \int_{P_i} f dz = \int_{\nu} f dz, \quad (2.2)$$

the last equality follows since $a_0 = b_0 = a, a_n = b_n = b$ and the integrals over ν_i occur in pairs in the opposite direction and hence cancel away.

To complete the proof, we have to only say that, by compactness of $[0, 1]$, there are finitely many points $0 = t_1 < \dots < t_k = 1$ such that any two consecutive paths $\omega_i := \omega_{t_i}$ are 'very close' to each other.

So, one can ask: can one modify H so that ω_t are all piecewise smooth? The general answer is 'yes' but then we are getting into deeper trouble. Instead, we see a short cut here.

We carry out every thing as described above except the steps (2.1) and (2.2), which may not make sense because the intermediary the intermediary paths ω_i may not be piecewise smooth. So, we abandon them and replace them by line segments joining their endpoints, so each P_i is now a piecewise smooth contour and hence (2.1), (2.2) are valid. ♠

Corollary 2.1.1 Let γ be a null-homotopic contour in Ω . Then for every

holomorphic function f on Ω , we have

$$\int_{\gamma} f dz = 0.$$

Definition 2.1.2 Let $\Omega \subset \mathbb{C}$ be a domain. We say Ω is simply connected, if every closed contour in Ω is null-homotopic in Ω .

Remark 2.1.1 The entire plane is simply connected. Indeed any convex domain in \mathbb{C} is simply connected. Simply connectedness is a topological invariant property, i.e., if X and Y are two spaces which are homeomorphic to each other then one of them is simply connected iff other is. Thus any domain which is homeomorphic to a convex domain is simply connected. At this stage we do not know any other way to see more examples of simply connected domains. Neither we have any tools to test whether a given domain is simply connected or not. The above corollary fills this gap to certain extent. Let restate it as:

Theorem 2.1.2 Cauchy's Theorem: Homotopy Version *Let Ω be a simply connected domain. Then for every closed contour γ in Ω and every holomorphic function f , $\int_{\gamma} f dz = 0$.*

Remark 2.1.2 Thus, if we find one holomorphic function f on Ω and one closed contour in Ω such that $\int_{\omega} f(z) dz \neq 0$, then we know that Ω is not simply connected. Thus $\mathbb{C} \setminus \{0\}$ is not simply connected because $1/z$ is holomorphic on it and its integral on the unit circle is $2\pi i$. Indeed, with this method, we are sure that given any domain Ω and a finite subset A , the domain $\Omega \setminus A$ is not simply connected.

We still do not know any method to prove that given Ω is simply connected, when we fail to find such a function and a closed contour.

Theorem 2.1.3 *Let Ω be a domain in \mathbb{C} . Consider the following statements:*
(i) Ω is simply connected.

(ii) For every closed contour γ in Ω and every holomorphic function f , $\int_{\gamma} f dz = 0$.

(iii) Every holomorphic function in Ω has a primitive.

(iv) Every holomorphic function on Ω which never vanishes on Ω has a holomorphic logarithm, i.e., there exists a holomorphic function $g : \Omega \rightarrow \mathbb{C}$ such that $\exp(g(z)) = f(z)$, $z \in \Omega$.

(v) For every point $a \in \mathbb{C} \setminus \Omega$, and every closed contour γ in Ω , we have, $\int_{\gamma} \frac{dz}{z-a} = 0$.

We have (i) \implies (ii) \implies (iii) \implies (iv) \implies (v).

Proof: (i) \implies (ii) This is the statement of 2.1.2.

(ii) \implies (iii) This is the statement of the primitive existence theorem for each fixed holomorphic function f .

(iii) \implies (iv) Apply (ii) to $h = f'/f$ to obtain g such that $g' = f'/f$. Then

$$(\exp(-g)f)' = -\exp(-g)g'f + \exp(-g)f' = \exp(-g)(-f' + f') = 0.$$

Therefore $k\exp(g) = f$ for some constant $k \neq 0$. By choosing g such that $\exp(g)$ and f coincide at a point, we get $\exp(g) = f$.

(iv) \implies (v) is obvious. ♠

Remark 2.1.3 Indeed it is true that all these statements are equivalent. Obviously the difficulty is in proving (v) \implies (i) or for that matter in proving (ii) \implies (i). We shall not be able to do this in this chapter. On the other hand, we shall now launch a programme which will enable us to prove (v) \implies (ii). On the way, we shall also learn a lot of other related things. Classically, and in many books even today, simply connectivity is defined by condition (v). Then the statement '(v) implies (ii)' is known as Homology form of Cauchy's theorem. The property (v) can be termed as 'homological simple connectivity' as compared to homotopical one above.

Thus the integral occurring in (v) assumes a special role. We shall take up the study of this quantity in some detail in the next section, before turning our attention to actual proof of homology form of Cauchy's theorem.

2.2 Winding Number

Lemma 2.2.1 Let γ be a closed contour not passing through a given point z_0 . Then the integral $\omega = \int_{\gamma} \frac{dz}{z - z_0}$ is an integer multiple of $2\pi i$.

Proof: Enough to prove that $e^{\omega} = 1$. Define

$$\alpha(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0} ds; \quad g(t) = e^{-\alpha(t)}(\gamma(t) - z_0); \quad a \leq t \leq b.$$

Since γ is continuous and differentiable except at finitely many points, so is g . Moreover, wherever g is differentiable, we have

$$\begin{aligned} g'(t) &= -e^{-\alpha(t)}\alpha'(t)(\gamma(t) - z_0) + e^{-\alpha(t)}\gamma'(t) \\ &= e^{-\alpha(t)}(-\gamma'(t) + \gamma'(t)) = 0. \end{aligned}$$

Therefore, $g(t) = g(a) = \gamma(a) - z_0$, for all $t \in [a, b]$ and hence,

$$e^{\alpha(t)} = \frac{\gamma(t) - z_0}{\gamma(a) - z_0},$$

for all $t \in [a, b]$. Since $\gamma(a) = \gamma(b)$, it now follows that $e^{\omega} = e^{\alpha(b)} = e^{\alpha(a)} = 1$.

♠

Definition 2.2.1 Let ω be a closed contour not passing through a point z_0 .

Put

$$\int_{\omega} \frac{dz}{z - z_0} = 2\pi i m.$$

Then the number m is called the winding number of the closed contour ω around the point z_0 and is denoted by $\eta(\omega, z_0)$. Thus

$$\eta(\omega, z_0) := \frac{1}{2\pi i} \int_{\omega} \frac{dz}{z - z_0}.$$

Remark 2.2.1 In order to understand the concept of winding number let us examine it a little closely.

1. Take $z_0 = 0$ and ω to be any circle around 0. Then we have seen that

$$\int_{\omega} \frac{dz}{z} = 2\pi i.$$

In other words, $\eta(\omega, 0) = 1$. So we can say that ω winds around 0 exactly once and this coincides with our geometric intuition.

2. Now let ω be any simple closed contour contained in the interior of an open disc in the upper half plane. Since $1/z$ is holomorphic in that disc, it follows that $\int_{\omega} \frac{dz}{z} = 0$. That means $\eta(\omega, 0) = 0$. Hence in this case, we see that the winding number is zero which again conforms with our geometric understanding.
3. More generally, if ω is contained in a disc, then for all points z outside this disc, we have $\eta(\omega, z) = 0$. This is a simple consequence of Cauchy's theorem for discs or by simply observing that $\frac{1}{z-a}$ has a primitive on the disc. Once again this conforms with our general understanding that such a contour does not go around z .
4. Let us now consider the curve $\omega(t) = e^{2\pi i n t}$, defined on the interval $[0, 1]$ for some integer n . This curve traces the unit circle n -times in the counter clockwise direction. This tallies with the computation of

$$\int_{\omega} \frac{dz}{z} = 2\pi i n.$$

5. It is obvious that $\eta(\omega, z) = -\eta((\omega)^{-1}, z)$. Moreover, if $\omega = \omega_1 \cdot \omega_2$, then

$$\eta(\omega, z) = \eta(\omega_1, z) + \eta(\omega_2, z).$$

6. By continuity of the integrated function, it follows that $z \mapsto \eta(\omega, z)$ is a continuous function on $\mathbb{C} \setminus Im(\omega)$. Being an integer valued continuous function, it must be locally constant. Therefore, it is a constant function on each path connected subset of $\mathbb{C} \setminus Im(\omega)$.
7. Enclosing ω in a large circle C and taking a point z_0 outside C , it follows from (3) that $\eta(\omega, z_0) = 0$. Hence, it follows that $\eta(\omega, z) = 0$ for all points z in the unbounded component of $\mathbb{C} \setminus Im(\omega)$. (Moreover, any unbounded component should intersect $\mathbb{C} \setminus B_r(0)$ for large r and hence there is only one unbounded component of $\mathbb{C} \setminus \omega$.)

Theorem 2.2.1 *Cauchy's Integral Formula (over simply connected domains):*
 Let Ω be a simply connected domain on which f is holomorphic. Then for any point $z \in \Omega$ and any closed contour ω in Ω not passing through z_0 , we have

$$\eta(\omega; z_0)f(z_0) = \frac{1}{2\pi i} \int_{\omega} \frac{f(z)}{z - z_0} dz. \quad (2.3)$$

Proof: Consider the function

$$F(z) = \frac{f(z) - f(z_0)}{z - z_0}$$

which holomorphic throughout $\Omega \setminus \{z_0\}$ and has a removable singularity at z_0 .

Therefore

$$0 = \int_{\omega} F(z) dz = \int_{\omega} \frac{f(z)}{z - z_0} - 2\pi i \eta(\omega, z_0).$$

this completes the proof. ♠

Remark 2.2.2 The following special case of theorem 2.2.1 is of utmost importance: Assume that for some component D_1 of $\mathbb{C} \setminus Im(\omega)$, we have, $\eta(\omega, w) = 1, \forall w \in D_1$. Then on D_1 , f itself is represented by

$$f(w) = \frac{1}{2\pi i} \int_{\omega} \frac{f(z) dz}{z - w}, \quad \forall w \in D_1. \quad (2.4)$$

In particular, the holomorphic function f is completely determined on this region by the value of f on ω and we have

$$f(z) = \frac{1}{2\pi} \int_{\omega} \frac{f(\xi)}{\xi - z} d\xi. \quad (2.5)$$

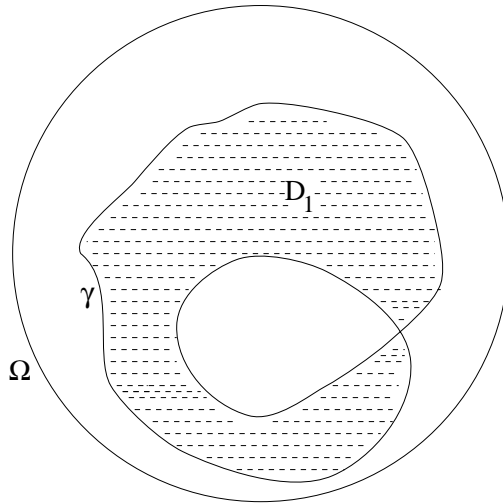


Fig. 1

Example 2.2.1 Let us find the value of

$$\int_{|z|=1} \frac{e^{az}}{z} dz.$$

Observe that e^{az} is holomorphic on the entire plane \mathbb{C} and the curve ω defining the unit circle has the property $\eta(\omega, 0) = 1$. Hence, by (2.4), the given integral is equal to $2\pi i e^0 = 2\pi i$.

Example 2.2.2 As a simple minded application of theorem 2.2.1, let us prove the non existence of certain roots. Assume that D is a domain which contains a closed contour $\omega : [a, b] \rightarrow \mathbb{C}$, such that $\eta(\omega, 0)$ is odd. Then we claim that there does not exist any holomorphic function $g : D \rightarrow$

\mathbb{C} such that $g^2(z) = z, z \in D$. Let us assume on the contrary. Then by differentiating, we get, $2g(z)g'(z) = 1, z \in D$. Now,

$$\eta(g \circ \omega, 0) = \frac{1}{2\pi i} \int_{g \circ \omega} \frac{dw}{w} = \frac{1}{2\pi i} \int_a^b \frac{g'(\omega(t))\omega'(t)}{g(\omega(t))} dt = \frac{1}{4\pi i} \int_a^b \frac{\omega'(t)}{\omega(t)} dt = \frac{\eta(\omega, 0)}{2}.$$

This means that $\eta(\omega, 0)$ is even which is absurd. Similar statements will be true for other roots also, viz, we do not have a well defined n^{th} root of $z - z_0$ in any domain that contains a closed contour ω such that $\eta(\omega, z_0)$ is not divisible by n .

Example 2.2.3 Let us now consider the function $f(z) = 1 - z^2$ and study the question when and where there is a holomorphic single valued branch g of the square root of f i.e., $g^2 = f$. Observe that $z = \pm 1$ are the zeros of f and hence if these points are included in the region then there would be trouble: By differentiating the identity $g^2 = f$ we obtain $2g(z)g'(z) = f'(z) = -2z$. This is impossible since, at $z = \pm 1$, the L.H.S. = 0 and R.H.S. = ∓ 2 . So the region on which we expect to find g should not contain ± 1 .

Next assume that Ω contains a small circle C around 1, say, contained in a punctured disc $\Delta' := B_\epsilon(1) \setminus \{1\}$ around 1. Restricting our attention to Δ' , observe that there is a holomorphic branch of the square root of $1 + z$ say h defined all over $B_\epsilon(1)$. Clearly $h(z) \neq 0$ here and hence $\phi = g/h$ will then be a holomorphic function on $\Delta' \cap \Omega$ such that $\phi^2 = 1 - z$. This contradicts our observation in the example 2.2.2.

By symmetry, we conclude that Ω cannot contain any circle which encloses only one of the points $-1, 1$.

Finally, suppose that both ± 1 are in the same connected component of $\mathbb{C} \setminus \Omega$. Then for all cycles ω in Ω , both ± 1 will be in the same connected component of $\mathbb{C} \setminus \text{im } \omega$ and hence $\eta(\omega, 1) = \eta(\omega, -1)$. For instance, take $\Omega = \mathbb{C} \setminus [-1, 1]$. Then for any circle C with center 0 and radius > 1 , $\eta(C, 1) = \eta(C, -1) = 1$.

We shall now see that the square root of f exists. Consider the fit $T(z) = \frac{1-z}{1+z}$. This maps $\mathbb{C} \setminus [-1, 1]$ onto $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$, on which we can choose a well defined branch of the square root function. This amounts to say that we have a holomorphic function $h : \mathbb{C} \setminus [-1, 1] \rightarrow \mathbb{C}$ such that $h(z)^2 = \frac{1-z}{1+z}$. Now consider $g(z) = h(z)(1+z)$. Then $g(z)^2 = f(z)$, as required.

In fact, $\Omega (= \mathbb{C} \setminus [-1, 1])$ happens to be a maximal region on which $1-z^2$ has a well defined square root. This follows from our earlier observation that any such region on which g exists cannot contain a circle which encloses only one of the two points $-1, 1$.

Finally, observe that, in place of $[-1, 1]$, if we had any arc joining -1 and 1 , the image of such an arc under T would be an arc from 0 to ∞ and hence on the complement of it, square-root would still exist. Also, the above discussion holds verbatim to the function $(z-a)(z-b)$ for any $a \neq b \in \mathbb{C}$. You can also modify this argument to construct other roots.

Here is a question then :

Ex. Prove or disprove that $f(z) = 1-z^2$ does not have a well defined logarithm in $\mathbb{C} \setminus [-1, 1]$.

Definition 2.2.2 Let Ω be a domain ω be a closed contour in it. We say ω is null homologous in Ω if for every point $a \in \mathbb{C} \setminus \Omega$, the winding number vanishes, $\eta(\omega, a) = 0$. If every closed contour in Ω is null-homologous in Ω , we say Ω is *homologically simply connected*. Two closed contours ω_1, ω_2 in Ω are homologous in Ω if $\eta(\omega_1, a) = \eta(\omega_2, a)$ for all $a \in \mathbb{C} \setminus \Omega$.

Theorem 2.2.2 *If ω is null-homotopic in Ω then it is null homologous in Ω . If two contours are path homotopic in Ω then they are homologous in Ω . A simply connected domain Ω is homologically simply connected.*

Proof: This is a direct consequence of theorem 2.1.1.

Remark 2.2.3 In view of (2.2.1.2.2.2), we shall give a sufficient condition for a contour to have winding number ± 1 around a point. This condition is quite a practical one in the sense that it is easy to verify in many concrete situations. In the statement of the lemma below, we have simply assumed that $z_0 = 0$. Of course, this does not diminish the generality of the result, as we can always perform a translation and choose the origin to be any given point. The result is important from application as well as theoretical point of view. *However, you may skip learning the proof of this for the time being and come to it later.*

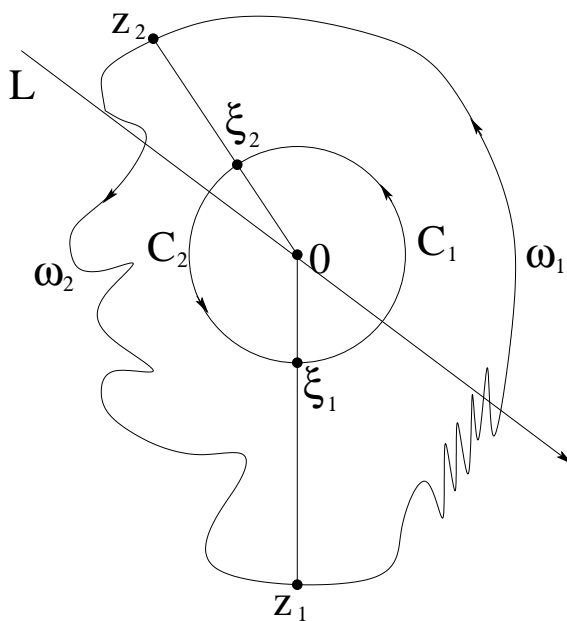


Fig. 2

Lemma 2.2.2 Let ω be a contour not passing through 0. Let z_1, z_2 be two distinct points on ω and let L be a directed line through 0 so that w_1 and w_2 are on the opposite sides of L . Denote the portion of the curve ω from z_1 to z_2 in the counter clockwise direction by ω_1 and the rest of the portion

by ω_2 so that we have $\omega = \omega_1 \cdot \omega_2$. Assume further that ω_1 does not meet the *negative ray* of L and ω_2 does not meet the *positive ray* of L . Then

$$\frac{1}{2\pi i} \int_{\omega} \frac{dz}{z} = \eta(\omega, 0) = \pm 1.$$

Proof: Let C be a circle around 0, not meeting ω and let ξ_1, ξ_2 be the points on C lying on the line segments $[0, z_1]$ and $[0, z_2]$ respectively. If C_1 and C_2 are the portion of the circle traced counter clockwise from ξ_1 to ξ_2 and from ξ_2 to ξ_1 respectively, it follows that C_1 does not meet the negative ray of L and C_2 does not meet the positive ray of L . Let

$$\tau_1 = [\xi_1, z_1] \cdot \omega_1 \cdot [z_2, \xi_2] \cdot (C_1)^{-1}; \quad \tau_2 = (C_2)^{-1} \cdot [\xi_2, z_2] \cdot \omega_2 \cdot [z_1, \xi_1].$$

Then it follows that τ_i are closed contours and

$$\eta(\tau_1, 0) + \eta(\tau_2, 0) = -\eta(C, 0) + \eta(\omega, 0).$$

On the other hand, since τ_1 does not meet the negative ray of L it follows that 0 is in the unbounded component of $\mathbb{C} \setminus \text{Im}(\tau_1)$ and hence as observed in remark 2.2.1.7, this implies that $\eta(\tau_1, 0) = 0$. For similar reason $\eta(\tau_2, 0) = 0$. Therefore,

$$\eta(\omega, 0) = \eta(C, 0) = 1.$$

If we had taken the other orientation on ω , we would have got $\eta(\omega, 0) = -1$. This completes the proof of the lemma. ♠

As an immediate application we have:

Theorem 2.2.3 *Let $\Omega \subset \mathbb{C}$ be a bounded convex region with a smooth boundary C oriented counter clockwise. Then*

$$\eta(C; a) = \begin{cases} 1 & a \in \Omega \\ 0, & a \in \mathbb{C} \setminus \bar{\Omega}. \end{cases}$$

In particular, this is true for any disc and any rectangle.

Proof: First consider $a \in \Omega$. Any line L through a , cuts C into two parts. Now for any two points z_1, z_2 on C lying on opposite sides of L , the hypothesis of the above lemma is easily verified. This gives the first part.

Now appeal to the fact that $\mathbb{C} \setminus C$ has two components, one of which is Ω and the other is $\mathbb{C} \setminus \bar{\Omega}$ which is unbounded. By remark 2.2.1.7, the second part follows. ♠

2.3 Homology Form of Cauchy's Theorem

Recall that while defining line integrals we first considered differentiable paths. Then, using the additivity property of the integral so obtained under subdivision of arcs, we could immediately generalize the definition of the integral over contours (which are, by definition, piecewise differentiable paths). We can now go one step further and allow our contours to have finitely many discontinuities also. (After all, recall that finitely many jump discontinuities do not cause any problem in the Riemann integration theory.) But then this is nothing but merely taking a finite number of contours γ_i together. Guided by the property that the integral over two non overlapping contours is the sum of the integrals over the two contours individually, and by the property that the integral over the inverse path is the negative of the integral, we now introduce a formal definition:

Definition 2.3.1 By a 'chain' we shall mean a finite formal sum $\sum_j n_j \gamma_j$ where n_j are any integers, and γ_j are contours. In this sum if each γ_j is a closed contour, then we call it a cycle. Observe that it does not hurt us if some of the integers n_j are zero. However, we do not generally write such terms in the summation. We can add two chains and rewrite the sum by 'collecting terms' if there are same contours occurring in the summation. The support of a chain is the set of all image points of all those γ_j for which n_j is not zero.

The following lemma is central in the proof of Cauchy's theorem. Primarily, instead of asserting that the formula holds for all contours it says there is a special one for which it holds. Even though we could do with a little weaker version of this lemma, we have chosen this form of the lemma, which can be used for other purposes later. It has its own importance having a certain topological content.

Lemma 2.3.1 Let U be an open subset of \mathbb{C} and K be a compact subset U . Then there exists a cycle ω in $U \setminus K$ such that

- (i) for all points $z \in K$, we have, $\eta(\omega, z) = 1$ and
- (ii) there exists an open subset U' such that $K \subset U' \subset U$ with the property that for all $z \in U'$ and for any holomorphic function f on U' we have,

$$f(z) = \frac{1}{2\pi i} \int_{\omega} \frac{f(\xi)}{\xi - z} d\xi \quad (2.6)$$

Proof: Since $\mathbb{C} \setminus U$ and K are disjoint closed sets and K is compact, we have,

$$\delta := d(\mathbb{C} \setminus U, K) = \inf\{|z_1 - z_2| : z_1 \in \mathbb{C} \setminus U, z_2 \in K\} > 0.$$

Choose $0 < \mu < \delta/3$. Raise a grid of horizontal and vertical lines with distance between consecutive parallel lines = μ . Let $\mathcal{R} = \{R_j\}$ denote the collection of all little squares belonging to this grid which are at a distance $\leq \delta/3$ from K . Since K is compact, this collection has only finitely many squares. We shall denote the contour obtained by tracing the boundary of a square R_j in the counter clockwise sense by ∂R_j . (It does not matter where you start off.)

Put $\omega' := \sum_j \partial R_j$ and $R = \cup_j R_j$.

Then clearly ω' is a cycle in U and $K \subset R$. Observe that ω' is a chain consisting of directed edges of squares in the collection \mathcal{R} . We delete each pair of edges which are opposite of each other occurring in ω' to obtain a

cycle ω . Clearly, ω is a cycle in U . Integrals over either of these cycles will be the same for all functions. (In particular the two cycles are homologous to each other.) Moreover, the support of ω does not intersect K at all. For, if any edge intersects K , then both the squares of the grid at this edge are in the collection \mathcal{R} and hence, the edge will occur twice, once in each direction, so gets deleted. This also shows that K is contained in the interior of R , which we set equal to U' .

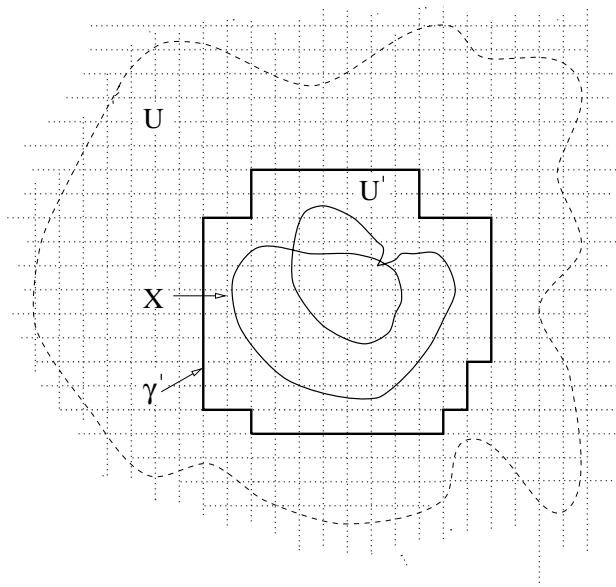


Fig. 4

Now, given any point z in K , since z does not lie on the support of ω , it follows that $\eta(\omega, z)$ makes sense. Also, every point of K belongs to one of the squares in \mathcal{R} . If $z \in \text{int } R_k$, then clearly $\eta(\partial R_j, z) = 1$ if $j = k$ and $= 0$ otherwise. (See theorem 2.2.3). Therefore, $\eta(\omega, z) = \eta(\omega', z) = 1$. Now the set $\cup_k \text{int } R_k$ is a dense open subset of U' . We know that winding number is locally a constant function. It follows that $\eta(\omega, z) = 1$, for all $z \in U'$.

To see the second part, let z be in the interior of one of the R_j , s . Then

$$\frac{1}{2\pi i} \int_{\partial R_k} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} f(z) & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Therefore,

$$f(z) = \frac{1}{2\pi i} \int_{\omega'} \frac{f(\xi)}{\xi - z} d\xi$$

for all points in $\cup_k \text{int } R_k$ which is a dense subset of U' . Since both sides of the above equation are continuous functions of z on U' , the validity of the equation (2.6) for all points of U' follows. ♠

We are now ready to prove the equivalence of (ii) and (v) of Theorem 2.1.3.

Theorem 2.3.1 Cauchy's Theorem: Homology Version: *Let Ω be a region in \mathbb{C} and γ be a cycle in Ω . Then the following conditions on γ are equivalent:*

- (i) $\int_{\gamma} f dz = 0$ for all holomorphic functions f on Ω .
- (ii) $\eta(\gamma, a) = 0$, for all $a \in \mathbb{C} \setminus \Omega$.

Proof: Suppose (i) holds. For any point $a \in \mathbb{C} \setminus \Omega$, the function $\frac{1}{z - a}$ is holomorphic on Ω and hence from (i), we obtain

$$\eta(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a} = 0.$$

This proves the statement: (i) \implies (ii).

To prove (ii) \implies (i), let A be the unbounded component of $\mathbb{C} \setminus \text{supp } \gamma$ and $K = \mathbb{C} \setminus A$. Since A is open K is closed. Indeed K is the union of all bounded components and $\text{supp } \gamma$ and hence is bounded. So it is compact. Take $U = \Omega \cup K$. Then U is open being the union of Ω and all the bounded components of $\mathbb{C} \setminus \text{supp } \gamma$. By the second part of the lemma 2.3.1, there exists

an open subset $U' \subset U$ which contains $\text{supp } \gamma$ and a cycle ω disjoint from K such that for all points $z \in U'$, we have,

$$f(z) = \frac{1}{2\pi i} \int_{\omega} \frac{f(\xi) d\xi}{\xi - z}.$$

On the other hand, $\text{supp } \omega \cap K = \emptyset \implies \text{supp } \omega \subset A \implies \eta(\gamma, \xi) = 0$ for all $\xi \in \text{supp } \omega$.

Therefore,

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} \left(\frac{1}{2\pi i} \int_{\omega} \frac{f(\xi)}{\xi - z} d\xi \right) dz \\ &= \int_{\omega} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{\xi - z} \right) f(\xi) d\xi = \int_{\omega} \eta(\gamma, \xi) f(\xi) d\xi = 0 \end{aligned}$$

This completes the proof of the theorem. ♠

Chapter 3

Convergence in Function Theory

3.1 Sequences of Holomorphic Functions

One of the most important results in the theory of convergence of functions is the majorant criterion of Weierstrass

Theorem 3.1.1 Weierstrass's Majorant Criterion: *Given a series of functions $\sum_n f_n$, suppose there is a convergent series $\sum_n M_n$ of positive terms such that $|f_n(x)| \leq M_n$ for all $x \in X$ and for all $n \gg 0$, then the series $\sum_n f_n$ is absolutely and uniformly convergent in X .*

Following this, we make some formal definitions: Let $f_n : X \rightarrow \mathbb{C}$, $n \geq 1$ be a sequence of functions.

Definition 3.1.1 We say the series $\sum_n f_n$ is *compactly convergent* in X if restricted to any compact subset of X , it is uniformly convergent.

Definition 3.1.2 The series $\sum f_n$ is said to be *normally convergent* in X if for every point $x \in X$, there exists a neighborhood U such that $\sum_n |f_n|_U < \infty$.

Recall that for each n , $|f_n|_U = \sup\{|f_n(z)| : z \in U\}$. Observe how Weierstrass's criterion has been adopted into a definition here: for a normally convergent series, the terms $|f_n|$ play the role of majorants, in the neighborhood U . It follows that every normally convergent series is locally uniformly convergent in X . Indeed, for series of continuous functions over domains in euclidean spaces (local compactness!), normal convergence is a convenient terminology for absolute local uniform convergence. In general, i.e., without continuity of functions (or local compactness of domains), normal convergence is slightly stronger than absolutely locally uniform convergence. When each f_n is continuous, it follows that the sum $\sum f_n$ is also continuous. Linear combination of normally convergent series is normally convergent. Also the same is true of Cauchy products of normally convergent series. In addition, because of the built-in absolute convergence, we have the following two results.

Theorem 3.1.2 *Every subseries of a normally convergent series is normally convergent.*

Theorem 3.1.3 Rearrangement Theorem: *Let $\sum f_n$ be a normally convergent series. Then for any bijection $\tau : \mathbb{N} \rightarrow \mathbb{N}$ the series $\sum f_{\tau(n)}$ is also normally convergent to the same sum.*

Proof: Let f be the sum $\sum f_n$. Let $x \in X$ and U be a neighborhood of x such that $\sum |f_n|_U < \infty$. By the rearrangement theorem for absolute convergent series of complex or (real) numbers, it follows that $\sum |f_{\tau(n)}|_U$ is convergent to $\sum_n |f_n|_U$. This means that $\sum f_{\tau(n)}$ is normally convergent. ♠

The following theorem, due to Weierstrass guarantees the holomorphicity of the limit under normal convergence. It also provides us the validity of term-by-term differentiation.

Theorem 3.1.4 Weierstrass' Convergence Theorem: *Suppose that we are given a sequence f_n of compactly convergent holomorphic functions in a*

domain D . Then the limit function f is holomorphic and the sequence $f_n^{(k)}$ compactly converges to $f^{(k)}$ on D for every positive integer k .

Proof: Observe that the limit function f is continuous in D due to uniform convergence on compact sets. Let γ be the boundary of a rectangle contained in D . Then the sequence f_n converges uniformly to f on γ . Hence it follows that

$$\lim_n \left(\int_{\gamma} f_n(z) dz \right) = \int_{\gamma} f(z) dz.$$

By Cauchy-Goursat theorem, each term on the lhs vanishes and hence rhs also vanishes. Hence, by Morera's theorem, it follows that f is holomorphic at $z_0 \in D$.

For the second part, to each closed ball $L = B_{2r}(z_0)$ contained D , put $K = B_r(z_0)$. First, we use Cauchy's integral formula to find M_r such that $|f_n^{(k)} - f^{(k)}|_K \leq M_r |f_n - f|_L$ for all n , as follows:

$$|f_n^{(k)}(z) - f^{(k)}(z)| = \left| \frac{k!}{2\pi i} \int_C \frac{f_n(\xi) - f(\xi)}{(\xi - z)^{k+1}} d\xi \right|$$

for all $z \in K$. So, we take $M_r = k!/r^k$. Since f_n uniformly converges to f on L , it follows that so does $f_n^{(k)}$ to $f^{(k)}$ on K . ♠

Example 3.1.1 As a typical example consider the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$. If $\Re(z) > 1 + \epsilon$, then $|n^z| = n^{|z|} > n^{1+\epsilon}$ and hence the series $\sum_1^{\infty} \frac{1}{n^{1+\epsilon}}$ is a majorant for the given series. This implies that $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is a normally convergent series and hence defines a holomorphic function $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ in the right-half plane $G_1 = \{z : \Re(z) > 1\}$. In fact, it is uniformly convergent in $\mathbb{G}_{1+\epsilon} = \{x + iy : x > 1 + \epsilon\}$. This function is called Riemann's zeta-function.

Remark 3.1.1 In the above theorem, one can start with a series which is normally convergent in D and then conclude similarly with normal convergence of the derived series. However, you may have learnt that a statement similar to that of the above theorem in the real case is false. A typical counter example is

$$f_n(x) = \frac{x}{1 + nx^2}$$

defined on the interval $(-1, 1)$. It is not difficult to show that $\{f_n\}$ compactly converges to the function f which is identically zero on the interval. But, $0 = f'(0) \neq \lim_{n \rightarrow \infty} f'_n(0) = 1$. (Why then does the sequence $f_n(z) = \frac{z}{1 + nz^2}$ not provide a counter example to the above theorem?)

Theorem 3.1.5 Weierstrass' Double Series Theorem. *Consider a sequence $f_m(z) = \sum_n a_{mn}z^n$ of convergent series in a disc B . Suppose the series $f(z) = \sum f_m(z)$ converges normally in B . Then for each n the series $b_n = \sum_m a_{mn}$ is convergent and f is represented in B by the convergent power series $f(z) = \sum_n b_n z^n$.*

Proof: Clearly, by the above theorem, f is holomorphic in B . Also $f^{(n)} = \sum_m f_m^{(n)}$ for all n . Moreover f is represented by the Taylor's series, $f(z) = \sum_n f^{(n)}(0)z^n/n!$. By substituting expressions for $f^{(n)}(0)$, we obtain the result.



Remark 3.1.2 As an illustration that the Cauchy product of two compactly convergent series need not be convergent, consider for each n , $f_n = g_n = (-1)^n/\sqrt{n+1}$, the constant function for all n . Clearly $\sum_n f_n = \sum_n g_n$ are convergent. But the modulus of the k^{th} term of the Cauchy product satisfies:

$$|h_k| = \sum_{m=0}^k [(m+1)(k-m+1)]^{-1/2} > \sum_{m=0}^k \frac{1}{k+1} = 1.$$

Hence the Cauchy product, $\sum h_k$ is not convergent.

We shall end this section with a celebrated result due to Hurwitz¹ concerning preservation of the zeros under compact convergence.

Theorem 3.1.6 Hurwitz: *Let f_n be a sequence of holomorphic functions compactly convergent to f in a region D . If f_n have no zeros in D then either $f \equiv 0$ or f has no zeros in D .*

Proof: Assuming that $f \not\equiv 0$, since the zeros of f are isolated, given $z_0 \in D$, we can choose $r > 0$ such that $A := \{z : 0 < |z - z_0| < r\} \subset D$, and $f(z)$ is holomorphic and not equal to zero on A . If C is the outer boundary of A , then $1/f_n(z)$ converges uniformly to $1/f(z)$ on C and since $f'_n(z)$ converges uniformly to $f'(z)$ on C , it follows that $f'_n(z)/f_n(z)$ converges uniformly to $f'(z)/f(z)$ on C . Hence, we have,

$$\int_C \frac{f'(z)}{f(z)} dz = \lim_n \int_C \frac{f'_n(z)}{f_n(z)} dz.$$

Since f_n have no zeros in D every term on rhs is zero. On the other hand, the lhs counts the number zeros of f inside C . Hence in particular, it follows that $f(z_0) \neq 0$. ♠

Corollary 3.1.1 In the situation of the above theorem, assume further that f_n is injective in D for all n sufficiently large. Then the limit function f is either a constant or injective.

Proof: Exercise.

For sharper results in this direction see the excellent book of Remmert p. 261-262.

¹Adolf Hurwitz from Zurich is known for his work on analytic functions and Cantor's set theory. He should not be confused with V. Hurwitz, who is the author of a famous book on dimension theory jointly with Wallman.

3.2 Convergence for Meromorphic Functions:

Let us first recall a definition.

Definition 3.2.1 A function f is called *meromorphic* in a domain D if there is a closed discrete subset $P(f)$ of D such that f is holomorphic in $D \setminus P(f)$ and has poles at each point of $P(f)$. Naturally the set $P(f)$ is called the *pole-set* of f .

It follows that $P(f)$ is always countable, since every discrete subset of \mathbb{C} is. Observe that, in the definition of a meromorphic function f , we allow the set $P(f)$ to be empty and thus all holomorphic functions on D are also meromorphic. Since we know that a meromorphic function tends to ∞ at its poles, we can think of them as continuous functions on D with values in the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by mapping each pole onto ∞ . Recall that if $z = z_0$ is a pole of f of order $n \geq 1$, then it is a pole of order $n + 1$ for f' . Thus if f is meromorphic, so is f' and $P(f) = P(f')$. Let us denote the set of all meromorphic functions on D by $\mathcal{M}(D)$ and the set of holomorphic functions on D by $\mathcal{H}(D)$. Observe that for each $f \in \mathcal{M}(D)$ which is not identically zero, we have $1/f \in \mathcal{M}(D)$. Thus, for any two $f, g \in \mathcal{H}(D)$, with $g \neq 0$, we have, $f/g \in \mathcal{M}(D)$. We shall prove soon that every element $h \in \mathcal{M}(D)$ can be expressed as $h = f/g$, as above, as a consequence of Weierstrass' theorem. [In algebraic terminology, this means $\mathcal{M}(D)$ is the quotient field of the integral domain $\mathcal{H}(D)$.]

What we are interested in doing in this section is to develop the theory of convergence for sequences of meromorphic functions. The basic difficulty here, as you might have guessed, is the presence of poles. However, since we expect and are ready to allow the limit also to be meromorphic, it is probably not all hopeless. Some sort of finiteness with respect to poles, in any case, seems inevitable and we demand that in any compact subset of D , only finitely many of the terms of the sequence involved have poles. (Below,

we shall directly deal with series, the case of sequences being similar and easier.) This leads us to the following cautiously adapted definition:

Definition 3.2.2 A series $\sum f_n$ of meromorphic functions in D is called *compactly convergent* in D if for every compact subset K of D there exists a number $m(K)$ such that

- (1) $n \geq m(K) \implies P(f_n) \cap K = \emptyset$ and
- (2) the series $\sum_{n \geq m(K)} f_n$ converges uniformly on K .

We say that the $\{f_n\}$ of meromorphic functions is *normally convergent* if for every compact subset K of D condition 1) holds and in place of 2) we have the stronger condition:

- 2') $\sum_{n \geq m(K)} |f_n|_K$ is convergent.

Condition (1) is called *pole dispersion condition*. Under this condition, the remaining summands in the series are pole free on K and hence in particular, continuous. Observe that, 1) implies that $\cup_n P(f_n)$ is discrete and closed in D . It is also clear that it is enough to demand that 1) holds for all closed discs inside D instead of for all compact subsets of D . The theorem below follows directly from the above definition.

Theorem 3.2.1 Let $\sum f_n$ be a series of meromorphic functions in D compactly (resp. normally) convergent in D . Then there exists a unique meromorphic function f on D with the following property:

For each open subset U of D and for each integer m such that $P(f_n) \cap U = \emptyset$, $\forall n \geq m$, the series $\sum_{n \geq m} f_n$ converges compactly (resp. normally) in U to a holomorphic function F_U on U such that

$$f|_U = f_0|_U + f_1|_U + \cdots + f_{m-1}|_U + F_U. \quad (3.1)$$

In particular, f is holomorphic in $D \setminus \cup_n P(f_n)$.

Proof: Let U be an open subset of D whose closure is compact. Let $m(K)$ be as in 1) for $K = \overline{U}$. It follows that $F_U = \sum_{n \geq m(\overline{U})} f_n$, is a holomorphic function on U . Thus (3.1) defines a meromorphic function on U . If V is another open set with compact closure, $m(\overline{V})$ is chosen similarly,

$$f|_V = f_0|_V + f_1|_V + \cdots + f_{m(\overline{V})-1}|_V + F_V,$$

say for definiteness, $m(\overline{U}) \leq m(\overline{V})$, then on $U \cap V$ we have, $F_U = f_{m(\overline{U})} + f_{m(\overline{U})+1} + \cdots + f_{m(\overline{V})-1} + F_V$ and hence, it follows that $f_U = f_V$. Since D can be covered by a family of open sets $\{U_\alpha\}$ with their closure compact, we may define $f : D \rightarrow \mathbb{C}$ by $f(z) = f_{U_\alpha}(z)$, $z \in U_\alpha$. The rest of the claims of the theorem are easily verified. ♠

We call f the sum of the series and write $f = \sum f_n$. We emphasize that whenever you come across with an infinite sum of meromorphic function, you should remember that the pole dispersion condition 1). It is not at all difficult to see that linear combinations of compactly (resp. normally) convergent series of meromorphic functions produce compactly (resp. normally) convergent series again. For normally convergent series, we even have the rearrangement theorem. Perhaps not so obvious is:

Theorem 3.2.2 Term-wise Differentiation Theorem: *Let $f_n \in \mathcal{M}(D)$ and let $\sum f_n = f$ be compactly (resp. normally) convergent. Then the term-wise differentiated series $\sum f'_n$ compactly (resp. normally) converges to f' .*

Proof: Let U be an open disc such that the closed disc $\overline{U} \subset D$. Choose m such that $P(f_n) \cap U = \emptyset$, for all $n \geq m$. Then $\sum_{n \geq m} f_n$ converges compactly (resp. normally) to a holomorphic function F in U such that (3.1) holds. We can apply the term-wise differentiation to this partial series and get $F' = \sum_{n \geq m} f'_n$ on U . Addition of first $m-1$ terms does not violate the nature of convergence. Thus, $\sum_n f'_n$ is compactly (resp. normally) convergent in D .

Also because of (3.1) its sum g satisfies,

$$g|U = f'_0|U + \cdots + f'_{m-1}|U + F' = (f|U)'$$

Since this is true for all such discs in D , we obtain, $g = f'$, in D . ♠

We would like to employ the above theoretical discussion to a practical situation. The theme is somewhat in the reverse order. *We begin with a meromorphic function and try to write it as a sum of the most simple meromorphic functions.* We shall keep our discussion for the domain $D = \mathbb{C}$. (The case of the general domain being not much different.) Consider for instance the case when f has finitely many poles b_1, \dots, b_k with its respective singular parts $P_j \left(\frac{1}{z - b_j} \right)$. We know that P_j are actually polynomials and we have $f(z) = \sum_j P_j \left(\frac{1}{z - b_j} \right) + g(z)$, where g is an entire function. This kind of representation has many advantages as it directly gives much information about the function. In the general case when f has infinitely many poles the sum $\sum_j P_j \left(\frac{1}{z - b_j} \right)$ may not be convergent in any sense. (For instance, take $P = \mathbb{N}$ and $P_j(z) = z$, $\forall j \in \mathbb{N}$.) This is where we need to have the theory of normal convergence discussed above. The following result, due to Mittag-Leffler² tells us that, if we choose the ‘singular portions’ appropriately, we can always get a representation as above for all meromorphic functions on \mathbb{C} . The proof that we present here is simpler than the original one and is due to Weierstrass.

Theorem 3.2.3 Mittag-Leffler I-Version. *Let $P = \{b_1, b_2, \dots\}$ be a (countable) discrete subset of \mathbb{C} . Let P_j be non zero polynomials without con-*

²Magnus G. Mittag-Leffler(1846-1927) was a Swedish mathematician, a most colorful personality, loved and respected by all. He was greatly influenced by Weierstrass in his approach. His main contribution is in the theory of functions. He played a great part in inspiring later research.

stant terms. Then there exists a meromorphic function f in \mathbb{C} with pole set $P(f) = P$ and with the singular parts $P_j \left(\frac{1}{z - b_j} \right)$. Moreover, if h is any meromorphic function with $P(h) = P$ and with the corresponding singular parts $P_j \left(\frac{1}{z - b_j} \right)$, then h can be written in the form

$$h(z) = \sum_j \left[P_j \left(\frac{1}{z - b_j} \right) - p_j(z) \right] + g(z), \quad (3.2)$$

where, g is an entire function and p_j are suitably chosen polynomial functions.

Proof: In this problem, if P is a finite set there is nothing to discuss. Also, we can at will, add (or delete) a finite number of points to (or from) P without changing the nature of the problem. In particular, without loss of generality we may and will assume that $0 \notin P$ and P is infinite. Consider the Taylor expansion of $P_j \left(\frac{1}{z - b_j} \right)$ around the origin and let p_j be the partial sum of this expansion say, up to degree n_j . The idea is to choose n_j sufficiently large to suit our purpose. Consider the remainder term $f_j(z) = P_j \left(\frac{1}{z - b_j} \right) - p_j(z)$. We know that the Taylor's series for $P_j \left(\frac{1}{z - b_j} \right)$ is uniformly convergent on $|z| \leq |b_j|/2$. Hence, we can choose n_j so that

$$|f_j(z)| < 2^{-j}, \quad \forall |z| \leq |b_j|/2, \quad \forall j. \quad (3.3)$$

Now given any compact set K (since P is discrete), there exists a natural number $m = m(K)$ such that $K \subseteq \{z : |z| \leq |b_m|/2\}$. Hence, we can find some constants c_K such that the series $c_K + \sum_{j \geq m(K)} 2^{-j}$ serves as a majorant for the series $\sum_j (f_j/K)$. Thus conditions 1) and 2) of normal convergence have been verified. It follows that the series $\sum_j f_j$ converges normally to a meromorphic function h with $P(h) = P$ with singular part at b_j equal to $P_j \left(\frac{1}{z - b_j} \right)$. This completes the first part of the theorem. For the second

part, we have only to observe that the function $g(z) = f(z) - h(z)$ is entire.



Example 3.2.1 Consider the case wherein a set $\{b_j\}$ has been given with the property that there exists an integer $k \geq 0$ such that $\sum_j |b_j|^{-k} = \infty$, and $\sum_j |b_j|^{-k-1} < \infty$. Let us take $P_j(z) = z$ for all j . We have,

$$\frac{1}{z - b_j} = -\frac{1}{b_j} \left(1 + \frac{z}{b_j} + \cdots + \frac{z^n}{b_j^n} + \cdots \right).$$

So, we take, $n_j = k - 1$ for all j , in the proof of the above theorem. Then

$$p_j(z) = \begin{cases} \frac{1}{b_j} + \frac{z}{b_j^2} + \cdots + \frac{z^{k-1}}{b_j^k} & \text{if } k > 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Hence as in the proof of the above theorem, we have,

$$f_j(z) = \frac{z^k}{b_j^{k+1}} + \frac{z^{k+1}}{b_j^{k+2}} + \cdots +$$

Now, using Weierstrass' double series theorem, it can be directly verified that $\sum f_j(z)$ is normally convergent.

Work out the same problem taking $P_j(z) = z^2$, for all j . (It turns out that we can take $n_j = k - 2$, if $k \geq 2$.)

3.3 Runge's Theorem

Let γ be a closed contour. We know that $\int_\gamma z^n dz = 0$ for all $n \geq 1$. Therefore $\int_\gamma p(z) dz = 0$ for all polynomials as well. If we wish to prove a result like $\int_\gamma f(z) dz = 0$ for all holomorphic functions, the natural question to ask is:

Q.1 Can we approximate a given holomorphic function by polynomials uniformly on given contours?

Of course, approximation results of this kind will have many other usefulness as well. Having asked such a question, we can as well consider the question in its own right. That allows us immediately to change the question to

Q. 2 Given a contiguous function f on a compact subset $K \subset \mathbb{C}$ when can we approximate it by holomorphic (polynomial) functions?

The natural norm to take is of course the supnorm on the given compact set K with respect to which we are taking approximations. Certain conditions on K seems to be inevitable. For instance, let K be the unit circle and $f(z) = 1/z$. Suppose there exists a sequence of polynomial p_n converging uniformly to f on the unit circle. By maximum modulus principle it would follow that the sequence p_n is uniformly Cauchy in the interior of the circle also. Hence it has limit defined all the disc. By Weierstrass's theorem, this limit is then holomorphic. Thus we have extended $1/z$ holomorphically all over the disc?!

This example tells precisely what could go wrong and surprisingly that is all that we have to take care, viz., we cannot expect to approximate an arbitrary continuous function f this way: f has to be holomorphic in a nbd of K . Even this is not enough. We may have to allow the approximating functions to be rational functions rather than polynomial functions. One of the versions of Runge's theorem actually handles this situation.

Let us denote the set of all functions $f : K \rightarrow \mathbb{C}$ which are holomorphic on a nbd of K by $\mathcal{H}(K)$.

Theorem 3.3.1 Runge's theorem *Let K be a compact set $\{U_j\}$ be the set of bounded components of $\mathbb{C} \setminus K$. Let B be a set such that $B \cap U_j \neq \emptyset$ for all j . Then any $f \in \mathcal{H}(K)$ can be approximated uniformly on K by rational functions whose pole set is contained in B .*

The proof of this has two essential parts. Lemma 2.3.1 allows us to represent f as an integral. Using this, we first obtain an approximation of f by rational functions whose pole set lies on the segments of the contour of integration. The next step is to 'shift' these poles away to inside the set B .

Lemma 3.3.1 *Let $g : K \times [a, b] \rightarrow \mathbb{C}$ be a continuous function where K is a compact set. Given $\epsilon > 0$, there exists a partition of the interval $[a, b]$ say $a = t_1 < t_2 < \dots < t_{n+1} = b$ such that for each $j = 1, 2, \dots, n - 1$*

$$|g(z, t) - g(z, t_j)| < \epsilon, \quad \forall z \in K \quad \& \quad t_j \leq t \leq t_{j+1}. \quad (3.4)$$

Proof: This follows easily by standard arguments using the uniform continuity.

Lemma 3.3.2 *Let $\gamma[a, b] \rightarrow U$ be a smooth curve. Given $\epsilon > 0$ there exist finitely many points z_j in the image of γ and a rational function P with poles at z_j such that*

$$\left| \frac{1}{2\pi} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi - P(z) \right| < \epsilon, \quad z \in K.$$

Proof: Let $L(\gamma)$ denote the length of the curve. Take

$$g(z, t) = \frac{f(\gamma(t))}{\gamma(t) - z}$$

and find a partition of the interval as in the previous lemma so that condition (3.4) holds, with ϵ replaced by $2\pi L\epsilon$.

Put

$$P(z) = \frac{1}{2\pi i} \sum_j^n \frac{f(\gamma(t_j))}{\gamma(t_j) - z} (\gamma(t_{j+1}) - \gamma(t_j)).$$

Then P has its poles $z_j = \gamma(t_j)$.

Also,

$$\begin{aligned}
 |f(z) - P(z)| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - P(z) \right| \\
 &= \frac{1}{2\pi} \left| \sum_{j=1}^n \int_{t_j}^{t_{j+1}} \left(\frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(t_j))}{\gamma(t_j) - z} \right) \gamma'(t) dt \right| \\
 &\leq \frac{1}{2\pi} \sum_{j=1}^n \int_{t_j}^{t_{j+1}} |g(z, t) - g(z, t_j)| |\gamma'(t)| dt \\
 &< \frac{\epsilon L(\gamma)}{2\pi} 2\pi L = \epsilon.
 \end{aligned}$$

♠

Given a holomorphic function $f : U \rightarrow \mathbb{C}$ where $K \subset U$, let ω, U' etc. be as in lemma 2.3.1:

$$f(z) = \frac{1}{2\pi} \int_{\omega} \frac{f(\xi)}{\xi - z} d\xi.$$

Since ω is a finite sum of its smooth parts, lemma 3.5 quickly yields:

Lemma 3.3.3 Given $\epsilon > 0$ there exist a rational function whose pole set is away from K such that

$$|f(z) - P(z)| < \epsilon, z \in K. \quad (3.5)$$

We now proceed to the next step of shifting these poles. Observe that all the poles involved in lemma 3.3.3 are simple poles.

Lemma 3.3.4 Pole Shifting Lemma: Suppose p, q are in the same component C of $\mathbb{C} \setminus K$. Then $\frac{1}{z - p}$ can be approximated on K by rational functions having pole at q alone. Also, if C happens to be the unbounded component then $\frac{1}{z - p}$ can be approximated by polynomials on K .

Runge's Theorem follows immediately from lemmas 3.3.3 and 3.3.4.

Proof: Even though the poles involved are simple, while 'shifting them' the simple poles may become poles of higher order as we shall see. This point should not cause any problem, because, if we can approximate $\frac{1}{z - p}$ by

rational functions whose pole set is contained in $\{q\}$ then all powers of $\frac{1}{z-p}$ can also be approximated in the same fashion. Therefore, it is enough to prove that $\frac{1}{z-p}$ can be approximated by rational functions whose pole is contained in $\{q\}$.

We consider the two cases: The component C is (i) bounded; (ii) unbounded.

First consider (i) Choose a path τ from p to q away from K . Let $\delta > 0$ be the distance of τ from K . Cover τ with a finite number of interlaced open balls $B_k, k = 1, \dots, n$, say, of radius $\delta/4$. Choose, $p = p_1, \dots, p_n = q$ on τ such that $p_k \in B_{k-1} \cap B_k, k = 2, \dots, n-1$. The idea is to approximate $\frac{1}{z-p_j}$ by a polynomial in $\frac{1}{z-p_{j+1}}$.

Thus the general case has been reduced to the special case, where we can assume that $|p-q| < \delta/4$. Thus for all $z \in K$, we have, $|p-q|/|z-q| < 3/4$. We now simply take the geometric expansion as follows:

$$\frac{1}{z-p} = \frac{1}{z-q} \left(1 + \frac{p-q}{z-q} + \dots + \left(\frac{p-q}{z-q} \right)^j + \dots \right).$$

Depending on $\epsilon > 0$ we have choose as many terms as permissible from the right hand side to get the required approximation.

This completes the proof, in the first case.

Now consider (ii) We first choose a point q in C far away from K . $|q| > 2|z|$ for all $z \in K$. We then proceed as in the first case to obtain an approximation of $\frac{1}{z-p}$ by a polynomial in $1/z-q$. Now it is enough to prove that $1/z-q$ can be approximated by a polynomial function.

$$\frac{1}{z-q} = -\frac{1}{q} \left(1 + \frac{z}{q} + \dots + \frac{z^j}{q^j} + \dots \right)$$

having radius of convergence $|q|$. In order that the expansion is valid all over K , it is therefore enough to choose q such that $|q| > 2|z|, z \in K$.

This completes the proof of the lemma and thereby the proof of Runge's theorem. ♠

Corollary 3.3.1 Let $\Omega \subset \mathbb{C}$ be a domain. Then the following statements are equivalent.

- (i) Ω is homologically simply connected
- (ii) $\hat{\mathbb{C}} \setminus \Omega$ is connected.
- (iii) for every holomorphic function on Ω , there exists a sequence of polynomials P_n which converges f in $\mathcal{H}(\Omega)$.

Proof: (i) \implies (ii) Suppose not. Let K be the union of all bounded components of $\hat{\mathbb{C}} \setminus \Omega$, and L be the unbounded component. Then $K \neq \emptyset$ is compact (being closed and bounded (why?)). Take $U = K \cup \Omega$. Then U is open. So, we can apply lemma 2.3.1 to conclude that there is a contour ω in Ω such that $\eta(\omega, a) = 1$ for all $a \in K$ which is a contradiction to (i).

(ii) \implies (i) This follows easily from remark 2.2.1.7.

(ii) \implies (iii) We can write Ω as a union of compact sets

$$K_1 \subset K_2 \subset K_3 \cdots$$

such that $\hat{\mathbb{C}} \setminus K_n$ is connected. Observe that a rational function whose pole set is $\{\infty\}$ is nothing but a polynomial. Therefore by choosing $B = \{\infty\}$ and applying Runge's theorem, we get a polynomial P_n such that $\|f - P_n\|_{K_n} < 1/n$. This is what is meant by saying that P_n converges to f in $H(G)$.

(iii) \implies (i) Taking $f = 1/z - a$ for some $a \in \mathbb{C} \setminus \Omega$, we see that $\eta(\omega, a)$ is the limit of the sequence $\{\frac{1}{2\pi i} \int_{\omega} P_n(z) dz\}$ each member being zero. Therefore $\eta(\omega, a) = 0$. This implies (i). ♠

Remark 3.3.1 Thus we have completed the proof that all the statements except (i) of theorem 2.1.3 are equivalent. The proof that (ii) \implies (i) involves yet another deep and landmark result, viz.,

Theorem 3.3.2 Riemann Mapping Theorem: *Every homologically simply connected domain in \mathbb{C} which is not the whole of \mathbb{C} is biholomorphic to the unit disc.*

Recall that biholomorphic mean that there is a bijective holomorphic mapping between the two spaces. It is easily seen that biholomorphic maps preserve homotopy and hence if one is simply connected the other one is. Since the unit disc is simply connected, it follows that homologically simply connected domains are simply connected. There are several proofs of this celebrated RMT but none of them is easy. Surprisingly, other than through RMT there is no known proof of the fact that a homologically simply connected domain is simply connected.

Bibliography

- [A] Ahlfors, V., *Complex Analysis*, III edition. McGraw-Hill Book Company 1979.
- [B] Bell, E.T., *Men of Mathematics*, Pelican Book Vol I, II (1937)
- [Caj] Cajori, F., *A History of Mathematical Notations*, The Open Court Publishing Co. Chicago, Illinois.
- [Car] Cartan, H., *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Addison-Wesley Publishing Company Inc., 1963.
- [C-B] Churchill, R.V. and Brown, J. W., *Complex Variables and Applications*, IV edition, Mc-Graw-Hill Int. Editions, 1984.
- [Con] Conway, J. B., *Functions of One Complex Variable*, II edition, GTM 11, Springer-Verlag (1973)
- [Cop] Copson, E.T., *An Introduction to the Theory of Functions of a Complex Variable*, Oxford, Clarendon Press, 1962.
- [Cox] Cox, David A., *Gauss and the Arithmetic Mean*, Notices of Amer. Math. Soc. March 1985.
- [G] Gamow, G., *One, Two . . . Infinity*, A Mentor Book, The New American Library, (1947).

- [H] Hille, E., *Analytic Function Theory, Vol. I, II*, Blaisdell Publishing Company 1959.
- [K] Kreyszig, Edwin, *Advanced Engineering Mathematics*, (6th Edition), Wiley Eastern Limited, New Delhi.
- [N] Narasimhan, R., *Complex Analysis in One Variable*, Birkäuser 1985.
- [P-L] Polya, G. and Latta, G., *Complex Variables*, John-Wiley & Sons Inc., 1974.
- [Re] Remmert, R., *Theory of Complex Functions*, GTM-122, Springer-Verlag 1980.
- [Ru] Rudin, W. *Real and Complex Analysis*, McGraw Hill Book Co. 1986 (II edition).
- [S-G] Sansonne, J. and Gerretsen, J., *Lectures on the Theory of Functions of a Complex Variable*, Vol I-II, P. Noordhoff-Groningen 1960.
- [Sh] Shastri A. R. *An introduction to Complex Analysis*, MacMillan India, 1999.
- [Si] Silvermann, R.A., *Introductory Complex Analysis*, Dover Publications Inc., 1972.
- [St] Struik D. J., *A Concise History of Mathematics*, IV edition, Dover Publication New York 1987.
- [Su] Sury, B., *Fundamental Theorem of Algebra*, Bulletin of the Bombay Math. Coll. Vol. 10. No. 1-2 pp. 53-57.
- [T] Tutte, W.T., *On Elementary Calculus and the Good Formula*, Journ. Comb. Th. (B) 18, pp. 97-137. (1975)

Exercises on Chapter 1

Diff. Calculus of 2-Variables

1. Suppose $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Show that each of the following functions on \mathbb{R}^2 are continuous.
 - (i) $(x, y) \mapsto f(x) + g(y)$; (ii) $(x, y) \mapsto f(x)g(y)$;
 - (iii) $(x, y) \mapsto \max\{f(x), g(y)\}$; (iv) $(x, y) \mapsto \min\{f(x), g(y)\}$.

2. Use the above exercise, if necessary to show that $f(x, y) = x + y$ and $g(x, y) = xy$ are continuous functions on \mathbb{R}^2 . Deduce that every polynomial function in two variables is continuous. Can you generalize this?

3. Express the definition of $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ in terms of polar coordinates and use it to analyze for the following functions:
 - (i) $f(x, y) = \frac{x^3 - xy^2}{x^2 + y^2}$; (ii) $g(x, y) = \tan^{-1} \left(\frac{|x| + |y|}{x^2 + y^2} \right)$;
 - (iii) $h(x, y) = \frac{y^2}{x^2 + y^2}$.

4. Examine the following functions for continuity at $(0, 0)$. The expressions below give the value of the function at $(x, y) \neq (0, 0)$. At $(0, 0)$ you are free to take any value you like.
 - (i) $\frac{x^3y}{x^2 - y^2}$; (ii) $\frac{x^2y}{x^2 + y^2}$; (iii) $xy \frac{x^2 - y^2}{x^2 + y^2}$;
 - (iv) $||x| - |y|| - |x| - |y|$; (v) $\frac{\sin^2(x + y)}{|x| + |y|}$.

5. Examine each of the following functions for continuity.
 - (i) $f(x, y) = \begin{cases} \frac{y}{|y|} \sqrt{x^2 + y^2}, & y \neq 0, \\ 0 & y = 0. \end{cases}$

$$(ii) \ g(x, y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & x \neq 0, y \neq 0; \\ x \sin \frac{1}{x}, & x \neq 0, y = 0; \\ y \sin \frac{1}{y}, & x = 0, y \neq 0; \\ 0 & x = 0, y = 0. \end{cases}$$

6. Let $f : B_r(0) \rightarrow \mathbb{R}$ be some function where $B_r(0)$ is the open disc of radius r and centre 0 in \mathbb{R}^2 . Assume that the two limits

$$l(y) = \lim_{x \rightarrow 0} f(x, y); \quad r(x) = \lim_{y \rightarrow 0} f(x, y) \quad (3.6)$$

for all sufficiently small y and for all sufficiently small x respectively exist. Assume further that the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = L$ also exists. Then show that the iterated limits

$$\lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x, y)], \quad \lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x, y)] \quad (3.7)$$

both exist and are equal to l .

7. Put $f(x, y) = \frac{x - y}{x + y}$, for $(x, y) \neq (0, 0)$. Show that the two iterated limits (3.7) exist but are not equal. Conclude that the limit $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

8. Put $f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2}$, $(x, y) \neq (0, 0)$. Show that the iterated limits (3.7) both exist at $(0, 0)$. Compute them. Show that the $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist.

9. Consider the function

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

- (a) Show that f is continuous and all the directional derivatives f of exist and are bounded.

- (b) For any \mathcal{C}^1 mapping $g : \mathbb{R} \rightarrow \mathbb{R}^2$ show that $f \circ g$ is a \mathcal{C}^1 - mapping.
 (c) Yet f is not differentiable at $(0, 0)$. [Hint: Use polar coordinates.]

10. Let f be holomorphic. Suppose $g : \mathbb{C} \rightarrow \mathbb{R}$ is a smooth map. Show that

$$\nabla^2(g \circ f)(z) = |f'(z)|^2 \nabla^2(g)(f(z))$$

Deduce that if g is harmonic then $g \circ f$ is harmonic.

Complex Differentiability

- Check for differentiability of the following functions directly from the definition at $z = 0$ and explain what goes wrong in case it is going wrong:
 (a) $z^2 + z + 1$; (b) $z^{1/2}$.
- Use chain rule, product rule and (1.2) to obtain the quotient rule.
- Use quotient rule, chain rule etc. to find the derivatives of
 (a) $\frac{z}{1+z}$; (b) $\frac{z^{1/2}}{z^2+z+1}$, wherever they exist.
- Let $f : U \rightarrow \mathbb{C}$ be a complex differentiable function, where U is a convex open subset of \mathbb{C} . Then show that f is a constant on U .
- Write down all possible expressions for the Cauchy derivative of a complex function $f = u + v$ in terms of partial derivatives of u and v .
- Verify that the functions $\Re z$, $\Im z$, and \bar{z} do not satisfy CR equations.
- Show that the function $f(z) = z\Re z$ is complex differentiable only at $z = 0$ and find $f'(0)$. How about $z\Im z$ and $z|z|$?
- Show that $|z|$ is Frechet differentiable everywhere except at $z = 0$. Can you say the same thing about complex differentiability?

9. Derive the following polar co-ordinate form of CR equations for $f = u(r, \theta) + w(r, \theta)$:

$$\boxed{ru_r = v_\theta; \quad rv_r = -u_\theta.} \quad (3.8)$$

10. Let f be a holomorphic function on an open disc such that its image is contained in a line (or a circle or a parabola). Show that f is a constant.
11. Try to generalize the statement above.
12. Show that $f(x, y) = \sqrt{|xy|}$ is continuous and has partial derivatives which satisfy C-R equation at $(0, 0)$, yet f is not complex differentiable at $(0, 0)$. Does this contradict Looman-Menchoff theorem?
13. Show that $u(x, y) = 2x(1 - y) + 1$ is harmonic and find the holomorphic function f with $\Re(f) = u$ and its conjugate by integration method as well as by formal method.
14. Find the holomorphic function f so that its real part is given by

$$\Re(f)(x, y) = \frac{\sin x + \cos x}{\cos^2 x + \sinh^2 y}.$$

15. Show that a real polynomial $p(X, Y)$ is harmonic iff all of its homogeneous parts are harmonic.
16. Discuss the harmonicity of a real homogeneous polynomial $p(X, Y)$ of degree 1 or 2.
17. Let $p(X, Y) = aX^3 + bX^2Y + cXY^2 + dY^3$ be a homogeneous polynomial of degree 3 in X, Y over reals. i.e., $a, b, c, d \in \mathbb{R}$. Find the most general condition under which p is harmonic and find a conjugate.

18. Let $n \geq 4$. Determine the necessary and sufficient conditions on the coefficients of $p(X, Y) = \sum_{j+k=n, j \neq k} a_{jk} X^j Y^k$, for p to be harmonic. Use this to give a direct proof of the fact that if $p(X, Y)$ is harmonic then $2p(z/2, z/2i)$ has real part equal to $p(X, Y)$. (This explains to some extent the mysterious looking arguments that we came across in this section.)
19. Show that sum of two harmonic functions is harmonic and scalar multiple of a harmonic function is harmonic.
20. Show that for a smooth function u on a domain D in \mathbb{C} , we have,

$$\nabla^2 u = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} u = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} u.$$

21. **Weak Branch Lemma:** Let $f : \Omega_1 \rightarrow \Omega_2$ be a holomorphic function, $g : \Omega_2 \rightarrow \Omega_1$ be a continuous function such that $f \circ g(w) = w$, $\forall w \in \Omega_2$. Suppose $w_0 \in \Omega_2$ is such that $f'(z_0) \neq 0$, where $z_0 = g(w_0)$. Then g is \mathbb{C} -differentiable at w_0 , with $g'(w_0) = (f'(z_0))^{-1}$.

Exercises on Chapter 2

Cauchy's Theory

1. Find the length of the following curves:
 - (i) The line segment joining 0 and $1 + i$.
 - (ii) The hypo-cycloid given by: $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $0 \leq \theta \leq 2\pi$, where, $a > 0$ is a fixed number.
 - (iii)* The perimeter of an ellipse with major and minor axes of size a and b respectively. (*Caution: this is rather a difficult problem.*)
2. Compute $\int_{|z|=\rho} x dz$, where $|z| = \rho$ is the circle of radius ρ around 0 taken in the counter clockwise sense.

3. Compute $\int_{|z|=\rho} z^n dz$, for all integers n . Use this to compute the integral in Ex. 2 in a different way, by writing $x = (z + \bar{z})/2 = (z + \rho^2 z^{-1})/2$.
4. Begin with the algebraic identity

$$1 + t + t^2 + \cdots + t^n = \frac{1 - t^{n+1}}{1 - t}$$

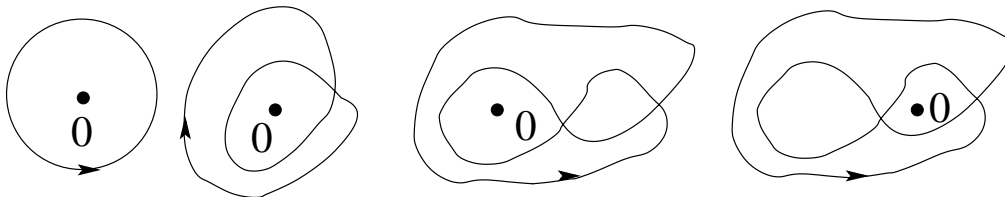
substitute $t = \frac{z-a}{\xi-a}$, and obtain, for any holomorphic function defined on the disc $|z - a| \leq r$, the following **Taylor's Expansion**:

$$f(z) = f(a) + f'(a)(z - a) + \cdots + \frac{f^{(n)}(a)}{n!}(z - a)^n \quad (3.9)$$

$$+ \frac{1}{2\pi i} \int_{|\xi-a|=r} \frac{f(\xi)}{(\xi - a)^{n+1}(\xi - z)} d\xi. \quad (3.10)$$

5. Show that $\int_0^\pi e^{a \cos \theta} \cos(a \sin \theta) d\theta = \pi$.
6. Compute $\int_{|z|=2} (z^2 + 1)^{-1} dz$.
7. Find the value of $\int_{|z|=\rho} \frac{|dz|}{|z - a|^2}$, assuming that $|a| \neq \rho$.
8. *Given a domain D , a point $a \in \mathbb{C}$ and a closed contour ω in D such that $\eta(\omega, a) \neq 0$, show that there exists a closed contour ω_1 in D such that $\eta(\omega_1, a) = 1$.
9. Show that if f is an entire function such that $|f(z)| \leq k|z^n|$ for some constant k and some positive integer n , then f is a polynomial function. Also what can you say about the degree of this polynomial?
10. Given a holomorphic function f such that $|f(z)| \leq 1, \forall |z| \leq 1$, find an upper bound for $|f^{(n)}(z)|$ in the discs $|z| < r$, for $0 < r < 1$. Conclude that $|f^{(n)}(z)| \leq n!$, for $|z| \leq 1$.

11. Let f be a holomorphic function on the unit disc such that $|f(z)| < (1 - |z|)^{-1}$. Use Cauchy's estimate to find an upper bound for $|f^{(n)}(0)|$.
12. Let f be holomorphic on $B_1(0)$ and suppose that $|f(z)| \leq 1$ in $B_1(0)$. Show that $|f'(0)| \leq 1$.
13. Find $\int_{\omega} z^n (z - 1)^{-1} dz$, where $\omega(t) = 1 + e^{2\pi it}$, $0 \leq t \leq 1$, and $n \geq 1$.
14. Compute $\int_{\omega} (z + z^{-1}) dz$, where ω is the unit circle traced in the counter clockwise sense.
15. Prove the following generalization of Cauchy's integral formula (2.5) :
Let f be a holomorphic function in a simply connected domain Ω , ω be any closed contour in Ω and $z_0 \in \Omega \setminus Im(\omega)$. Then for all $n \geq 1$, we have,
$$\int_{\omega} \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} \eta(\omega, z_0) f^{(n)}(z_0).$$
16. Let ω be some closed contour, p be a point not on ω , and L be an infinite ray beginning at p and intersecting ω in exactly n points at each of these points 'crossing' it over. Show that $\eta(\omega, a) \equiv n \pmod{2}$. Give a recipe to determine the actual value of $\eta(\omega, a)$ from this consideration.
17. Evaluate $\int_{\omega} (e^z - e^{-z}) z^{-4} dz$, where ω is one of the closed contour drawn below:



18. State and prove generalizations of Argument Principle.
19. Prove holomorphicity of Elementary symmetric functions.
20. Obtain an integral formula for f^{-1} in a nbd of a point where a holomorphic function f has non vanishing first derivative.

Exercises on Chapter 3

Function Theory

1. Let $f_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$, $Z_n := \{z : f_n(z) = 0\}$ and let $\tau_n = d(0, Z_n)$, be the distance between 0 and the set Z_n . Show that $\tau_n \rightarrow \infty$ as $n \rightarrow \infty$.
2. Find a meromorphic function f with $P(f) = \mathbb{Z}$ and principal parts $P_j(z)$ given by (i) z ; (ii) jz for all $j \in \mathbb{Z}$.
3. As in the example (3.2.1) above, if $P = \{j^p : j \in \mathbb{N}\}$, where $p > 0$ is a fixed real number, determine $k = k_j$. Do the same for $P = \{e^j : j \in \mathbb{N}\}$.
4. Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be any connected open set. Show that there exist compact sets $K_1 \subset K_2 \subset \cdots$ such that
 - (i) $\cup_r K_r = \Omega$
 - (ii) $\Omega \setminus K_r$ is connected $\forall r$.