

**Algebraic Topology**  
**Problem Set**  
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In the following a map means a continuous map unless otherwise stated, and  $I = [0, 1]$ .

1. A topological space  $X$  is contractible if and only if for any space  $Y$  two maps of  $Y$  into  $X$  are homotopic.
2. Show that a contractible space is pathwise connected.
3. Show that a space  $X$  is contractible if and only if it is homotopically equivalent to a point.
4. Let  $X$  be a path connected space. Then,  $X$  is simply connected if and only if every map of the unit circle  $S^1$  into  $X$  extends to a map of the closed unit disc  $E^2$  into  $X$ .
5. Let  $CX = X \times I / X \times \{0\}$ , be the cone on  $X$ . Regard  $X \subset CX$  via  $x \rightarrow (x, 1)$ . Show that  $f : X \rightarrow Y$  is homotopically trivial if and only if  $f$  extends to  $\bar{f} : CX \rightarrow Y$ .
6. Let  $f, g : S^n \rightarrow S^n$  be maps such that for all  $x \in S^n$ ,  $f(x)$  and  $g(x)$  are not antipodal. Show  $f \simeq g$ . If in addition there is  $x_0 \in X$  such that  $f(x_0) = g(x_0)$ , show  $f \simeq g \text{ rel } x_0$ .
7. Let  $X$  be pathwise connected, and suppose every  $f : S^1 \rightarrow X$  is homotopically trivial but not necessarily by a homotopy leaving the base point fixed. Show  $\prod_1(X, x_0) = 0$ .
8. Show that a discrete normal subgroup of a connected topological group is central.
9. Suppose the space  $X$  is the union of two open sets  $U$  and  $V$ , such that  $U \cap V$  is non-empty and pathwise connected, and  $U, V$  are each simply connected. Then, prove that  $X$  is also simply connected. (This is a special case of Van Kampen's theorem).
10. The  $n$ -sphere  $S^n$  is simply connected for  $n \geq 2$ .

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11. Let  $E \xrightarrow{p} X$  be a covering space of  $X$ . Show that if  $E$  is pathwise connected then all the fibres have the same cardinality.
12. Let  $E \xrightarrow{p} X$  be a covering space of  $X$  with  $E$  connected and locally path connected. Let  $N$  be the normalizer of  $P_*(\prod_1, (E, e_0))$  in  $\prod_1(X, x_0)$ , where  $p(e_0) = x_0$ . Then obtain a homomorphism of  $N$  onto  $G$  with kernel  $p_*(\prod_1(E, e_0))$ , where  $G$  is the group of covering transformations.
13. A covering space,  $E \xrightarrow{p} X$  is said to be a normal or regular covering space if  $p_*(\prod_1(X, e_0))$  is a normal subgroup of  $\prod_1(X, x_0)$  where  $p(e_0) = x_0$ .  
Let  $G$  be the group of covering transformations.  
Then prove the following:  
 $E \xrightarrow{p} X$  is a normal covering space if and only if  $G$  separates transitively on the fibre  $p^{-1}(x_0)$ .
14. Let  $E$  be a connected and locally path connected space and let  $G$  be a group of homeomorphisms of  $E$  which separates properly discontinuously (i.e. for any  $e \in E$ , there is an open neighborhood  $V$  such that  $V \cap gV = \emptyset$  for all  $g \neq 1$  in  $G$ ). Let  $X = E/G$  be the space of orbits,  $p : E \rightarrow X$  the map sending any  $e$  onto its orbit  $Ge$ . Show that:  
i)  $E \xrightarrow{p} X$  is a covering space  
ii)  $G$  is its group of covering transformation  
iii)  $p_*(\prod_1, (E, e_0))$  is a normal subgroup of  $\prod_1(X, x_0)$  for all  $e_0 \in E$ .
15. Show that  $\prod_1(\mathbb{R}P^n), n \geq 2$ , is generated by the composition  $pg$  where  $g : I \rightarrow S^n$  is any continuous map satisfying  $g(0) = -g(1)$  and  $p : S^n \rightarrow \mathbb{R}P^n$  is the covering map.  
In the following all spaces are assumed to be connected and locally path connected, unless otherwise stated.

16. Consider the following commutative diagram:

$$\begin{array}{ccc}
 & & (E, e_0) \\
 & \nearrow \tilde{f} & \downarrow p \\
 (Y, y_0) & \xrightarrow{f} & (X, x_0)
 \end{array}
 \quad p\tilde{f} = f$$

show that if  $f$  and  $p$  are covering maps then  $\tilde{f}$  is also a covering map.

17. Let  $E \xrightarrow{p} X$  be a covering space, where  $X$  is connected and locally path connected, but  $E$  is not connected. Let  $C$  be a connected (hence path - connected) component of  $E$ . Then show that  $p|_C : C \rightarrow X$  is a covering space.
18. Prove  $SO(3)$  is not homotopically equivalent to  $S^1 \times S^2$ .
19. Show that  $SU(2)$  is simply connected. What is the fundamental group of  $U(2)$ ?
20. Compute the fundamental group of  $GL(2, \mathbb{C})$ .
21. Consider the usual covering map  $p : \mathbb{R}^2 \rightarrow S^1 \times S^1$ . Describe the group of covering transformations.
22. Let  $p : X \rightarrow Y$  and  $q : Y \rightarrow Z$  be covering maps. Show that if  $q^{-1}(z)$  is finite for each  $z \in Z$ , then  $q \circ p : X \rightarrow Z$  is a covering map. Give an example to show that this conclusion might fail if  $q^{-1}(z)$  is not finite.