Problems on Potential Theory - II

(1) Show that $v(z)$ has a logarithmic pole at ζ if and only if $v(z) = \log|f(z)|$ for some function $f(z)$ that is meromorphic near ζ and has a simple pole at ζ.

(2) Let D be a bounded domain with smooth boundary, and let $u(z)$ be a smooth function on $\overline{D}$ such that $u(z) = 0$ on ∂D. Show that

$$\int\int_D u(z)\Delta u(z)\, dx\, dy \leq 0$$

with strict inequality unless $u(z) = 0$ for all $z \in D$.

(3) Let D be a bounded domain with smooth boundary, and let λ be a real number. Suppose there is a nonzero smooth function $u(z)$ on $\overline{D}$ such that $\Delta u = \lambda u$ on D and $u = 0$ on ∂D. Show that $\lambda < 0$.

(4) Let A be the annulus $\{a < |z| < b\}$. Solve the Dirichlet problem in A with boundary data $h(p) = \alpha$ for $|p| = a$ and $h(p) = \beta$ for $|p| = b$, where α, β are constants.

(5) Let D be a bounded domain in the plane whose boundary consists of two disjoint simple closed real analytic curves E_0, E_1. By solving the Dirichlet problem for the boundary data $h(z) = 0$ on E_0 and $h(z) = 1$ on E_1, show that D is biholomorphic to an annulus.

(6) Show that for the unit disc, the Green's function is given by

$$g(z, \zeta) = -\log\left|\frac{z-\zeta}{1-z\overline{\zeta}}\right|$$

where z, ζ are in the unit disc.

(7) Let D be a bounded domain with analytic boundary, and let $g(z, \zeta)$ be the Green's function for D with pole at ζ. If $u(z)$ is a smooth function on $\overline{D}$, then show that

$$u(\zeta) = -\frac{1}{2\pi}\int_{\partial D} u(z)\frac{\partial g}{\partial n}\, ds - \frac{1}{2\pi}\int_D g(z,\zeta)\Delta u(z)\, dx\, dy.$$

(8) Suppose that $\zeta \in D_0 \subset D$, and suppose that Green's function $g(z)$ for D with pole at ζ exists. Show that Green's function $g_0(z)$ for D_0 with pole at ζ exists and satisfies $g_0 \leq g$.

(9) Compute the Green's function for the upper half plane and the horizontal strip given by $\{z = x + iy : |y| < \pi/2\}$.

(10) Show that every domain in the complex plane is the increasing union of bounded domains each having analytic boundary.

(11) Let D be a domain in the complex plane. Show that Green's function for D with a pole at $\zeta \in D$ is the upper envelope of the family of functions $u(z)$ on D such that $u(z) + \log|z - \zeta|$ is subharmonic on D, and $u(z) = 0$ off a compact subset of D.

(12) Let $\{u_n\}$ be a sequence of positive harmonic functions on a Riemann surfaces R that converges uniformly on some coordinate disk. Show that the sequence converges uniformly on

compact subsets of R.

(13) Let $f : R \to S$ be a holomorphic mapping between Riemann surfaces R, S. Suppose that Green's function exists for S. Show that Green's function exists for R and that $g_R(p,q) \leq g_S(f(p), f(q))$.

(14) Let R be a Riemann surface for which Green's function does not exist. Let B denote a closed coordinate disk on R. Show that if $u(p)$ is a subharmonic function on $R \setminus B$ such that $u(p)$ is bounded above and $\limsup u(p) \leq 0$ as $p \to \partial B$, $p \in R \setminus B$, then $u(p) \leq 0$ for all $R \setminus B$.

(15) Show that Green's function for R exists if and only if there is a nonconstant subharmonic function on R that is bounded above.