Problems on Potential Theory - I

Notation: In what follows, Ω will be a domain in the complex plane and $D(p, r)$ the open disc of radius $r > 0$ centered at $p \in \mathbf{C}$.

(1) Let $h : \Omega \to \mathbf{R}$ be such that both h, h^2 are harmonic. Show that h is a constant.

(2) Let $h : \Omega \to \mathbf{R}$ be a harmonic function with the property that hg is harmonic for every harmonic function g on Ω. Show that h is a constant.

(3) Show that the Laplacian in polar coordinates is given by

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

with the standard identification $x = r\cos\theta$ and $y = r\sin\theta$. Using this, determine all the radial harmonic functions on the punctured unit disc. Are there any non-constant radial harmonic functions on the unit disc?

(4) Let $\Omega = D(0, 1) \setminus \{0\}$ and h a harmonic function on Ω. Using the fact that the function $h_x - ih_y$ is holomorphic on Ω show that

$$h(z) = \operatorname{Re}\left(\sum_{-\infty}^{\infty} a_n z^n\right) + b\log|z|$$

for $z \in \Omega$. (The discussion in Ahlfors' book on pages 162 - 165 will be useful here)

(5) Let u_n be a sequence of non-negative harmonic functions in Ω such that for some $p \in \Omega$, the series $\sum_{n\geq 1} u_n(p)$ converges. Show that the function $u(z) = \sum_{n\geq 1} u_n(z)$ is harmonic in Ω.

(6) Let u_n be a sequence of harmonic functions on Ω that converges locally uniformly to a (harmonic) function $u(z)$. Show that the sequence of partial derivatives $(u_n)_x, (u_n)_y$ also converges locally uniformly to u_x, u_y respectively. Generalize this to sequences of higher order mixed partials of u_n.

(7) Show that if a harmonic function on the complex plane is bounded above, then it reduces to a constant.

(8) Show that there are no non-constant harmonic functions on the Riemann sphere.

(9) Show that if h is a positive harmonic function on $D(0, r)$, then the absolute value of the gradient of h at the origin is bounded above by $\frac{2}{r}h(0)$.

(10) If h is a harmonic function on Ω then show that h^2 is subharmonic on Ω.

(11) Prove that if $\zeta \in \mathbf{C}$ and $r > 0$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|re^{it} - \zeta|\,dt$$

equals $\log|\zeta|$ if $r \leq |\zeta|$ else $\log r$ if $r > |\zeta|$. Using this show that

$$u(z) = \sum_{n\geq 1} 2^{-n} \log|z - 2^{-n}|$$

is subharmonic on the complex plane and that u is discontinuous at the origin.

(12) Let $u_1, u_2, \ldots, u_n$ be subharmonic on Ω such that their sum $u_1 + u_2 + \ldots + u_n$ attains a maximum on Ω. Show that each function u_j is harmonic on Ω.

(13) Let $u > 0, v > 0$ be given functions on Ω such that $\log u, \log v$ are both subharmonic functions on Ω. Show that $\log(u + v)$ is also subharmonic.

(14) Let a, b, c, d be non-negative, subharmonic functions on Ω with the property that the 2×2 matrix A formed by them by putting them in the 'usual' places is orthogonal for every $z \in \Omega$. Show that all four functions are constants.

(15) Let $\phi : \partial D(0,1) \to \mathbf{C}$ be defined by $\phi(\zeta) = \overline{\zeta}$. Show that there is no function f holomorphic on $D(0,1)$ and continuous on the closed unit disc with the property that $f = \phi$ on $\partial D(0,1)$. This shows that the holomorphic version of the Dirichlet problem may have no solution.

(16) Show that the Poisson kernel on the unit disc is given by

$$P(re^{it}, e^{i\theta}) = \sum_{n\in\mathbf{Z}} r^{|n|} e^{in(t-\theta)}$$

where $r < 1$ and t, θ vary between 0 and 2π.

(17) Let D be a proper subdomain of the Riemann sphere and let $p_0 \in \partial D \setminus \{\infty\}$ and suppose that there exists $p_1 \neq p_0$ such that the line segment $[p_0, p_1]$ does not meet D. Show that the function

$$b(z) = -\mathrm{Re}\left(\frac{z - p_0}{z - p_1}\right)^{1/3}$$

for $z \in D$ is a barrier at p_0.

(18) Solve the problem: find u on the closed unit disc such that $\Delta u = -xy$ on $D(0,1)$ and $u = 0$ on the unit circle. (Observe that if $v = -xy(x^2 + y^2)/12$, then the Laplacian of v equals $-xy$.)

(19) Let f be a bounded continuous function on the imaginary axis. Define u on the right half plane as

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x f(it)}{x^2 + (y - t)^2}\, dt.$$

where $z = x + iy$ and $x > 0$. Show that u is a bounded harmonic function on the right half plane such that for $c \in \mathbf{R}$, $\lim_{z\to ic} u(z) = f(ic)$ where $\mathrm{Re}\, z > 0$.

(20) Suppose that $u(x, y)$ is harmonic in a neighbourhood of the closed unit disc and that $u(x, y) = 2x^2 - y^2$ on the unit circle. Show that $u(x, y) \geq 2x^2 - y^2$ on the closed unit disc. Also find the value of $u(0, 0)$.

(21) Let $v : D(0,1) \to \mathbf{R}$ be a continuous subharmonic function. Assume that

$$\int_0^{2\pi} v(re^{it})\, dt \leq 1$$

for all $r \in (0,1)$. Prove that there exists a harmonic function u defined on $D(0,1)$ such that $v(z) \leq u(z)$ for all $z \in D(0,1)$.